PDE Surface Generation with the Combined Closed and Non-Closed Form Solutions

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Abstract Partial differential equations (PDEs) combined with suitably chosen boundary conditions are effective in creating free form surfaces. In this paper, a fourth order partial differential equation and boundary conditions up to tangential continuity are introduced. The general solution is divided into a closed form solution and a non-closed form one leading to a mixed solution to the PDE. The obtained solution is applied to a number of surface modelling examples including glass shape design, vase surface creation and arbitrary surface representation.

Keywords surface generation, combined solution, fourth order partial differential equation, geometric modelling

1 Introduction

Surface generation is an important topic in computer graphics and computer aided design. A surface can be represented with explicit form \( z = f(x, y) \), implicit form \( f(x, y, z) = 0 \), and parametric form \( \mathbf{x} = (u, v, z(u, v)) \) where \( x, y, z \) are the positional functions, and \( u \) and \( v \) are parametric variables. Among them, parametric surfaces are the most popular in computer graphics, virtual reality and computer aided design\(^4\). Most parametric surfaces use control-point-based modelling methods, such as Bézier, B-splines and NURBS. In recent years, physics based modelling methods is also gaining popularity.

Terzopoulos and Fleischer introduced mechanical laws of continuous bodies and proposed physically-based models of deformable curves, surfaces and solids\(^2\). This method was further extended to consider viscoelasticity, plasticity and fracture\(^3\). Celniker and Gossard applied the finite element method to generate primitives that build continuous deformable shapes designed to support a new free-form modelling paradigm\(^5\).

In order to alleviate the computational burden of deforming virtual objects with the finite element method, Kang and Kak used a two-stage strategy: a coarse resolution for a 3D calculation of the deformation, and consequently a finer resolution for just the surface layers of the object for a better and smoother delineation of the object shape\(^6\). Göldükay and Özgüç implemented two different formulations, primal formulation and hybrid formulation into a system for the animation of deformable bodies based on elasticity theory\(^6\).

Qiu and Terzopoulos developed dynamic NURBS, a new free-form shape model that marries the elegant geometry of rational multivariate simplex splines with physical dynamics and has arbitrary domains, non-degeneracy for multi-sided surfaces and other important features\(^7\). Guan et al. derived the equations of motion for the deformable curves and surfaces from Lagrangian mechanics and solved the equations with the finite element method with the applications to smooth surface joining, n-sided patch construction, curve and surface fairing etc.\(^8\).

Generating surfaces using the solution to partial differential equations subjected to suitably defined boundary conditions can be regarded as a physics-based geometric modelling method. It was proposed about two decades ago\(^9\). In spite of its short history, it has been applied widely in computer graphics and product design. The research into this topic is becoming increasingly active.

Partial differential equations based geometric modelling defines a surface with a partial differential equation and a set of boundary conditions. Therefore, the choice of the partial differential equation and boundary conditions and the resolution of the PDE are important factors. Some methods, both analytical and numerical, have been developed to tackle this problem.

Following their Fourier series method\(^9\), Bloor and Wilson subsequently developed a spectral approximation solution\(^10\). Both are semi-analytical
methods to the resolution of PDEs in surface modelling. In addition, numerical methods have also
been investigated. Using B-spline to represent PDE surfaces, they presented a collocation method\cite{11}. Cheng et al. examined the finite difference solution of partial differential equations\cite{12}. Brown et al. employed the finite element PDE solution to generate and modify non-uniform B-spline surfaces\cite{13}. Li gave the basic theory\cite{14} and bi-
harmonic equations\cite{15} of boundary penalty finite element methods, discussed the super-convergence and stability, and applied this method to surface blending\cite{16}. Since the Fourier series method and spectral method are the most efficient, they have been used to develop the user interface for interactive surface modelling\cite{17,18}. These numerical methods, however, are usually not computationally efficient.

Fast creation of geometric objects is required in many interactive computer graphics applications such as computer animation and virtual environments. Therefore, how to solve PDEs for surface modelling remains an important issue.

In our previous work, we have proposed a general fourth order partial differential equation\cite{19,20}. In this paper, we discuss an efficient method for the resolution of these PDEs. The basic idea is to divide the boundary conditions into two groups: one corresponds to a closed form solution, and the other to a non-closed form solution. Both solutions will then be combined to provide a general solution to the PDE.

2 Mathematical Definitions

The partial differential equation widely used by Bloor and Wilson is a biharmonic elliptic equation which contains one parameter and has the form of \cite{9}:

\[
\left( \frac{\partial^2}{\partial u^2} + \alpha^2 \frac{\partial^2}{\partial v^2} \right) x = 0
\]  

(1)

where \( \alpha = [a_x \ a_y \ a_z]^T \) is a vector-valued shape parameter and \( x = [x(u,v) \ y(u,v) \ z(u,v)]^T \) represents the generated surface.

Since parameter \( \alpha \) in the equation has important influence on the surface shape, it can be used as a user handle for surface shape manipulation. Motivated by this observation, we proposed a more general fourth order PDE with three vector-valued shape parameters, which is given by \cite{19, 20}:

\[
\left( a \frac{\partial^4}{\partial u^4} + b \frac{\partial^4}{\partial u^2 \partial v^2} + c \frac{\partial^4}{\partial v^4} \right) x = 0
\]  

(2)

where \( a = [a_x \ a_y \ a_z]^T, \ b = [b_x \ b_y \ b_z]^T, \ c = [c_x \ c_y \ c_z]^T \).

Partial differential (2) is a more general form of PDE (1). When taking \( a_x = a_y = a_z = 1, \ b = 2\alpha^2 \) and \( c = \alpha^4 \), PDE (2) is reduced to PDE (1).

Boundary conditions usually consist of the positional and tangential functions of the surfaces at their boundary curves. For surfaces defined by two boundary curves, the boundary conditions can be written in the following form

\[
\begin{align*}
\mathbf{x}(u_0, v) &= \mathbf{g}_1(v) \\
\mathbf{x}(u_1, v) &= \mathbf{g}_2(v) \\
\frac{\partial \mathbf{x}(u_0, v)}{\partial u} &= \mathbf{g}_3(v) \\
\frac{\partial \mathbf{x}(u_1, v)}{\partial u} &= \mathbf{g}_4(v)
\end{align*}
\]  

(3)

where \( g_i(v) \ (i = 1, 2, 3, 4) \) are the known functions of the boundary curves and the first derivatives of the surface at the boundary curves.

Through the above mathematical definitions, the problem to create surfaces is transformed into the resolution of PDE (2) subject to boundary conditions (3). Various numerical methods such as those mentioned in the previous section can be used. But they are generally slow, as discussed earlier. In this paper, we will develop an analytical solution to speed up the surface generation process.

3 Resolution of PDE

Two factors decide whether a partial differential equation has a closed form solution or not. They are the vector-valued parameters in the partial differential equation and the boundary conditions. For example, the closed form solution of a PDE may exist if the boundary conditions consist of periodic functions. This is usually not the case for surface modelling. Thus a good approximation becomes necessary.

In order to obtain such an approximate analytical solution, the boundary curve functions and the first derivatives will be split into separate non-polynomial terms. The terms whose second and fourth derivatives are the same as the terms themselves may have a closed form solution, and the remaining terms will be solved using an approximate representation.

Now let us transfer boundary condition functions (3) into the following form
\[
\begin{align*}
\mathbf{x}(u_0, v) &= a_{10} + \sum_{i=1}^{l_1} a_{i1} \mathbf{g}_i(v) + \sum_{i=1}^{l_2} b_{i1} \mathbf{g}_i(v) \\
\mathbf{x}(u_1, v) &= a_{20} + \sum_{i=1}^{l_1} a_{i2} \mathbf{g}_i(v) + \sum_{i=1}^{l_2} b_{i2} \mathbf{g}_i(v) \\
\frac{\partial \mathbf{x}(u_0, v)}{\partial u} &= a_{30} + \sum_{i=1}^{l_1} a_{3i} \mathbf{g}_i(v) + \sum_{i=1}^{l_2} b_{3i} \mathbf{g}_i(v) \\
\frac{\partial \mathbf{x}(u_1, v)}{\partial u} &= a_{40} + \sum_{i=1}^{l_1} a_{4i} \mathbf{g}_i(v) + \sum_{i=1}^{l_2} b_{4i} \mathbf{g}_i(v)
\end{align*}
\]

where the first and second terms on the right-hand side of the above equations correspond to a closed-form solution, and the third term leads to a non-closed-form solution.

In (4), the following vector operator has been used to facilitate the description

\[
\mathbf{pq} = \{ p_x, q_x, p_y, q_y, p_z, q_z \}
\]

where \( p = \{ p_x, p_y, p_z \} \) and \( q = \{ q_x, q_y, q_z \} \) are two column vectors.

Substituting the first and second terms of (4) into PDE (2), and using the method of variable separation, the closed-form solution of the first and second terms on the right-hand side of (4) can be taken to be

\[
\mathbf{x}(u, v) = C_0(u) + \sum_{i=1}^{l_1} C_i(y) \mathbf{g}_i(v)
\]

where

\[
C_0(u) = c_{00} + c_{01} u + c_{02} u^2 + c_{03} u^3
\]

\( c_{0k} (k = 0, \ldots, 3) \) are unknowns. Depending on the combination of the shape parameters in (2), \( C_i(u) \) has two different forms which are given by

when \( 4a_x x - x < b_x^2, 4a_y y < b_y^2, \) and \( 4a_z c_z < b_z^2, \)

\[
C_i(u) = c_{i1} e^{t_{1i} u} + c_{i2} e^{t_{2i} u} + c_{i3} e^{t_{3i} u} + c_{i4} e^{t_{4i} u}
\]

and when \( 4a_x c_x = b_x^2, 4a_y c_y = b_y^2, \) and \( 4a_z c_z = b_z^2, \)

\[
C_i(u) = (c_{i1} + c_{i2} u) e^{t_{1i} u} + (c_{i3} + c_{i4} u) e^{t_{2i} u}
\]

where \( c_{ij} (j = 1, \ldots, 4) \) are unknowns, and \( t_{ij} (i = 1, 2, \ldots, M; j = 1, 2, 3, 4 \text{ or } j = 1, 2) \) are determined by both the shape parameters in (2) and the functions in the boundary conditions (4).

For the third term of (4), the closed-form solution does not exist. The approximate analytical solution can be given by

\[
\mathbf{x}(u, v) = \sum_{i=1}^{l_2} \mathbf{x}_i(u, v) = \sum_{i=1}^{l_2} D_i(u) \mathbf{g}_i(v)
\]

(10)

where \( D_i(u) \) are the trial functions, within which some of the unknown coefficients can be determined by substituting (10) into boundary conditions (4).

Choosing \( L \) collocation points within the resolution region and substituting the values of these collocation points into (10), then substituting (10) into PDE (2), the residual functions are given by

\[
R_i(u_i, v_i) = \left( a \frac{\partial^4}{\partial u^4} + b \frac{\partial^4}{\partial u^3 \partial v} + c \frac{\partial^4}{\partial u^2 \partial v^2} \right) \mathbf{x}_i(u_i, v_i)
\]

\( (i = 1, 2, \ldots, I_2; l = 1, 2, 3, \ldots, L) \).

(11)

Minimising the squared sum of the above residual functions \( R_i \) leads to the derivation of the following linear algebraic equations

\[
\mathbf{A}^T \mathbf{A} \mathbf{C} = \mathbf{A}^T \mathbf{B}
\]

(12)

and the resolution of these linear equations determines the remaining unknown constants. It is worth mentioning that from our experiments, satisfying accuracy can be achieved with only a small number of collocation points resulting in good computational efficiency.

In order to study its accuracy, in the following section, we will compare the proposed method with the analytical resolution method.

4 Comparison with Closed Form Solution

For the cases where closed form solutions are not achievable, numerical methods can be employed to solve partial differential equations. However, numerical methods are time-consuming and not suitable to real-time computer graphics applications.

With the method proposed in this paper, this problem can be well solved.

In the following, we will demonstrate the good precision of our proposed method by comparing it with closed form solution for a same surface modeling problem. The boundary conditions for this
example can be written as

\[
\begin{aligned}
  u &= 0 \\
  x &= -2 \left( 1 - \frac{v}{\pi} \right), \quad \frac{\partial x}{\partial u} = 0, \\
  y &= -\cos v, \quad \frac{\partial y}{\partial u} = 0, \\
  z &= h_1, \quad \frac{\partial z}{\partial u} = h'_1
\end{aligned}
\]

where \( h_0 \) and \( h_1 \) are positions of bottom and top boundary curves of the surface along its height, and \( h'_0 \) and \( h'_1 \) are gradients of the surface at its two boundary curves.

With the method developed in this paper, the solution of PDE (2) under the above boundary conditions can be written as

\[
\begin{aligned}
  x &= C_{s1}(u) + C_{s2}(u) \sin v \\
  y &= C_{c1}(u) \cos v \\
  z &= C_{c1}(u)
\end{aligned}
\]  

where \( C_{s1}(u) \) is determined by the weighted residual method and the rest by the closed form resolution method.

Taking \( a_x = a_y = a_z = 1, b_x = b_y = b_z = 18 \), \( c_x = c_y = c_z = 81 \), the closed form solution of PDE (2) subjected to the boundary conditions (13) has the form of

\[
\begin{aligned}
  x &= c_{x00} + c_{x01} u + c_{x02} u^2 + c_{x03} u^3 + \\
  &\quad (c_{x10} + c_{x11} u + c_{x12} u^2 + c_{x13} u^3)v + \\
  &\quad [(c_{x20} + c_{x21} u) e^{3u} + (c_{x22} + c_{x23} u) e^{-3u}] \sin v \\
  y &= \left[ (c_{y00} + c_{y01} u) e^{3u} + (c_{y02} + c_{y03} u) e^{-3u} \right] \cos v \\
  z &= c_{x00} + c_{x01} u + c_{x02} u^2 + c_{x03} u^3
\end{aligned}
\]  

(15)

where \( c_{xij}, c_{yij} \) and \( c_{aaj} \) (\( i = 0, 1, 2; j = 0, 1, 2, 3 \)) are unknown constants.

Setting \( h_0 = 0, h_1 = 1, h'_0 = h'_1 = 0 \) and with the above two methods, we generate the surface shown in Fig.1. In this figure, (a) is from the closed form solution, and (b) is from our proposed method.

Fig.1. Comparison between two different methods.

Observing both images, we could not find any differences. It indicates that our proposed method has high computational accuracy.

In addition, the proposed method is also very efficient. It only took less than \( 10^{-6} \) of a second to determine all the unknown constants on an 800 MHz PC.

5 Examples of Surface Generation

The above proposed method can be used to create various surfaces quickly. In the following, we give a number of examples to demonstrate the use of this method.

5.1 Glass Shape Design

Glasses are devices widely applied in our daily life. They can be divided into two types: one consists of only one patch, and the other are composed of two or more surface patches. For each patch, it is defined by two boundary curves, normally circles of different radii. For the purpose of illustration, we here only consider the glass consisting of one surface segment. The boundary curve functions and the first derivatives can be written as

\[
\begin{aligned}
  u &= 0 \\
  x &= 1.1 \sin v, \quad \frac{\partial x}{\partial u} = 2.5 \sin v, \\
  y &= 1.1 \cos v, \quad \frac{\partial y}{\partial u} = 2.5 \cos v, \\
  z &= 0, \quad \frac{\partial z}{\partial u} = -2
\end{aligned}
\]

(16)

Examining (16), it can be found that the second and fourth derivatives of all the boundary functions in (16) are the same as themselves. Therefore, the
closed form solutions for all the components of PDE (2) exist, which are given by

$$\begin{align*}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} c_{1x}(u) \\ 0 \\ c_{2y}(u) \end{bmatrix} \sin v + \\
&\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cos v.
\end{align*}$$

(17)

The unknown functions $c_{1x}(u)$, $c_{2y}(u)$ and $c_{0z}(u)$ in the above equation can be determined by substituting (17) into PDE (2). Then, substituting (17) into the boundary conditions (16), all the unknown constants can be obtained, and (17) can be employed to generate the glass surface.

The shape parameters in PDE (2) have a strong effect on the shape of the surface to be created. Here we can demonstrate this by changing their values. First, we set the shape parameters to: $a_x = a_y = 2$, $b_x = b_y = 5$, and $c_x = c_y = 3$ and the glass shape is depicted in Fig.2(a). Then keeping parameters $b$ and $c$ unchanged ($b_x = b_y = 5$ and $c_x = c_y = 1$) and changing parameter $a$ to: $a_x = a_y = 0.01$, the glass given in Fig.2(a) is transformed to Fig.2(b). Similarly, keeping parameters $a$ and $b$ unchanged and changing parameter $c$ to: $c_x = c_y = 1$, the glass in Fig.2(c) is produced. Finally, fixing $a$ and $c$, and only changing $b$ to $b_x = b_y = 50$, we obtain the surface shown in Fig.2(d). These images demonstrate clearly that all the three shape parameters have a strong influence on the shape of the generated surface.

5.2 Vase Surface Representation

Vases are a common decoration commodity in our daily life. Their upper openings may be round or pedal-like in shape, sometimes with other adjunct features. Here, we consider two examples of vase design. The first example is to create a common vase surface which has two round openings. And the second example is to generate a pedal-like vase.

The boundary conditions defining the vase surface of the first example have the form of

$$\begin{align*}
&u = 0: \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r_0 \cos v \\ n_0 \sin v \\ h_0 \end{bmatrix}, \\
&u = 1: \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r_1 \cos v \\ n_1 \sin v \\ h_1 \end{bmatrix},
\end{align*}$$

(18)

where $n_0$ and $a_1$ are the radii of the boundary circles of the vase, $h_0$ and $h_1$ are the positions of the two boundary circles along the height of the vase, and $r_0', r_1', h_0'$ and $h_1'$ are the gradient of the vase at the two boundary circles.

Fig.3. Surface generation of a common vase.

For the above boundary conditions, the solution of PDE (2) has the similar form to that of (17), ex-
cept that the \( \sin v \) and \( \cos v \) in the equation are swapped.

Taking the parameters in (18) to be \( r_0 = 0.4, r'_0 = 2.55, h_0 = 0, h'_0 = 8.75, r_1 = 0.3, r'_1 = 1.7, h_1 = 3.5 \) and \( h'_1 = 5.25 \), and setting the vector-valued parameters to \( a_x = a_y = 1, b_x = b_y = 30 \) and \( c_x = c_y = 20 \), the vase in Fig.3 is created.

For the second example, the boundary conditions defining the vase can be formulated as

\[
\begin{align*}
\begin{cases}
    u = 0 \\
    x = \cos 2\pi v + 0.15\sin 4\pi v + 0.15\sin 8\pi v, \\
    y = \sin 2\pi v + 0.15\cos 4\pi v + 0.15\cos 8\pi v, \\
    z = 3.6 - 0.4\sin 10\pi v,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial x}{\partial u} = -0.1\cos 2\pi v - 0.3\sin 4\pi v - 0.6\sin 8\pi v, \\
\frac{\partial y}{\partial u} = -0.1\sin 2\pi v - 0.3\cos 4\pi v - 0.6\cos 8\pi v, \\
\frac{\partial z}{\partial u} = -0.1 + 0.5\sin 10\pi v;
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    u = 1 \\
    x = 0.8\cos 2\pi v, \\
    y = 0.8\sin 2\pi v, \\
    z = 0,
\end{cases}
\end{align*}
\]

These boundary conditions also provide us with a closed form solution, which takes the form of

\[
\begin{align*}
\begin{cases}
    x(u, v) = C_0(u) + C_1(u)\sin 2\pi v + C_2(u)\sin 4\pi v + \\
    C_3(u)\sin 8\pi v + C_4(u)\sin 10\pi v + \\
    C_5(u)\cos 2\pi v + C_6(u)\cos 4\pi v + \\
    C_7(u)\cos 8\pi v.
\end{cases}
\end{align*}
\]

(20) into PDE (2) and the unknown constants can be determined from the boundary conditions (19). The vase generated with (20) is given in Fig.4.

5.3 Surface Defined by Arbitrary Curves

As discussed before, some functions in the boundary conditions do not lead to a closed form solution. For these cases, an approximate analytical solution has to be sought. Here, we are to create a surface defined by an arbitrary space curve and a circle together with the first derivatives. The two boundary curves and the first derivatives are taken to be

\[
\begin{align*}
\begin{cases}
    u = 0 \\
    x = 1 - 6.17v + 20v^2 - \\
    13.33v^3, \\
    y = -2.13v + 36.27v^2 - \\
    68.27v^3 + 34.13v^4, \\
    z = 2.5,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial x}{\partial u} = 0, \\
\frac{\partial y}{\partial u} = 0, \\
\frac{\partial z}{\partial u} = -10.1;
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    u = 1 \\
    x = 1.2 - 1.4\sin 2\pi v, \\
    y = 0.9 - 1.4\cos 2\pi v, \\
    z = 0,
\end{cases}
\end{align*}
\]

(21)

For the above boundary conditions, the function expressions can be decomposed into two parts. The first part represents the closed form solution of PDE (2) with 1, \( v, 2\pi v \) and \( \cos 2\pi v \) being its basic function elements. The second part represents the non-closed form solution with \( v^2, v^3 \) and \( v^4 \) being its basic function elements.

![Fig.4. Vase surface creation.](image)

Same as above, the unknown functions in the above equation can be obtained by substituting

![Fig.5. Arbitrary surface representation.](image)
For the first part, we can take the same treatment as above and obtain its closed form solution. For the second part, the corresponding $x$ and $y$ components can be determined with the above-mentioned weighted residual method. The summation of the closed form solution and the approximate solution represents a mixed solution of PDE (2) subject to boundary conditions (21). The surface produced with this mixed solution is shown in Fig. 4.

The resolution procedure of all the above examples took less than $10^{-6}$ second on an 800MHz PC. It is fast enough for interactive applications.

5.4 Complex Surface Defined by a Number of Surface Patches

The final example is to model a claw. In order to model an object as complex as a claw, we first divide the surface into a number of surface patches. For each surface patch, its boundary curves together with the tangential properties of the surface at these boundary curves are determined. Then, PDE (2) is solved subject to these boundary constraints and the solution is employed to produce the surface patch. Assembling all the surface patches, we obtained the surface of the claw as shown in Fig. 6.

![Fig.6. Surface modelling of a claw.](image)

6 Conclusions

An analytical method has been proposed in this paper to create various free-form surfaces. According to the form of boundary conditions, the solution to the partial differential equation is divided into two parts: one is closed form, and the other is non-closed form. The resolution of both parts is discussed. Their combination gives the general solution of the partial differential equation subjected to the given boundary conditions.

The closed form solution satisfies both the boundary conditions and the partial differential equation. Therefore, the computational accuracy depends on the non-closed form solution. In order to achieve high computing accuracy, we have presented an approximate analytical solution. By determining the unknown constants in the solution functions, boundary conditions are met and the error of the partial differential equations is minimized. In addition to accuracy, it is also computationally efficient.

Employing the proposed method, a number of surface generation tasks were carried out. They include wineglass surfaces, vase surfaces, an arbitrary surface with a split and a claw consisting of multiple surface patches.

References


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