

SURFACE REPRESENTATION USING SECOND, FOURTH AND MIXED ORDER PARTIAL DIFFERENTIAL EQUATIONS

Jian J Zhang and Lihua You

National Centre for Computer Animation
Bournemouth University
Dorset BH12 5BB
United Kingdom

ABSTRACT

Partial differential equations (PDEs) are powerful tools for the generation of free-form surfaces. In this paper, techniques of surface representation using PDEs of different orders are investigated. In order to investigate the real-time performance and capacity of surface generation based on the PDE method, the forms of three types of partial differential equations are put forward, which are the second, mixed and fourth order PDEs. The closed form solutions of these PEDs are derived. The advantages and disadvantages of each of them are discussed. A number of examples are given to demonstrate the use and effectiveness of the techniques.

Key words: free-form surface representation, partial differential equations, orders of PDE

INTRODUCTION

Free-form surface representation is one of the most important topics of computer graphics and computer-aided geometric design. To model a complex object in a graphical scene, three forms of representations are commonly used. There are polygonal representations, parametric surfaces and implicit surfaces. Among them, parametric surfaces, such as spline surfaces, are the most popular form adopted in many modelling and design software systems.

Surfaces created using the solutions of partial different equations also have their unique merits in surface and object representation. A solvable partial differential equation corresponds to a family of solutions. This property allows certain shape requirements to be met and to generate a family of free-form surfaces. In addition, adjustable parameters can be introduced which provide further degrees of freedom for shape modification. This characteristic is especially attractive for the design of a family of products whose shapes do not differ fundamentally.

At the beginning of this decade, this surface representation method was first proposed by Bloor et al. (Bloor 1990). The fourth order partial differential equation with one vector-valued parameter was used. This equation was applied to represent a number of designs, including the hull of a yacht (Bloor 1990), an engine inlet port (Bloor 1995), and an aircraft fuselage (1996).

Activated by the fact that the vector-valued parameter in the PDE can effectively affect the shape of the surfaces to be generated, You and Zhang proposed a more general fourth order PDE for the blending of primary surfaces (You 1999b) and the generation of free form surfaces (You 2000a). Instead of having one parameter, this new equation has three vector-valued parameters.

So far as virtual reality and computer animation are concerned, fast surface generation is crucially important. This is even more apparent when the graphical scenes of such applications become increasingly complicated. Although modelling quality should not be compromised, some sort of adaptive strategy as for how algorithms are chosen will certainly be constructive. Considering the limitations and performance of different PDEs, we will investigate surface generation mechanisms using PDEs of different orders. Generally speaking, the higher the order a PDE has, the more powerful and flexible it is for surface generation, but the poorer the computational performance it has. The second order PDEs whose solutions involve only half of the unknown constants of those of the fourth order PDEs, are clearly computationally more efficient. But the second order PDEs lack the ability of considering the tangential boundary conditions and therefore have difficulty in shape control. If quality and performance are both important, the weakness of the second order PDE may be improved by using mixed order partial differential

equations. In this paper, we will introduce the techniques of free-form surface creation using the second, mixed and fourth order partial differential equations. We will also discuss their advantages and disadvantages.

SURFACE GENERATION WITH THE SECOND ORDER PDES

The second order PDE we are proposing takes the form of

$$\mathbf{s} \frac{\partial^2 \mathbf{x}}{\partial u^2} + \mathbf{t} \frac{\partial^2 \mathbf{x}}{\partial v^2} = \mathbf{p}(u, v) \quad (1)$$

where $\mathbf{s} = [s_x \ s_y \ s_z]^T$ and $\mathbf{t} = [t_x \ t_y \ t_z]^T$ are the vector-valued parameters, $\mathbf{x} = [x \ y \ z]^T$ represents the 3D co-ordinates of the generated surface, and $\mathbf{p}(u, v) = [p_x(u, v) \ p_y(u, v) \ p_z(u, v)]^T$ denotes the vector-valued force functions.

The boundary conditions for the generated surface can be generally written as

$$\begin{aligned} u = 0 & \quad \mathbf{x} = \mathbf{g}_1(u) \\ u = 1 & \quad \mathbf{x} = \mathbf{g}_2(u) \end{aligned} \quad (2)$$

To demonstrate the use of this second order PDE (1) under boundary conditions (2), in the following, we will construct a vase-like object. Such a PDE may or may not have closed form solutions of the partial differential equations depending on the boundary conditions it is subject to. If a closed form solution exists, such a solution usually is the best in terms of resolution performance and computational accuracy. For those boundary conditions which prevent a closed form solution from being developed, the PDE will be solved using the series method (Zhang 2000) combined with the weighted residual techniques (You 2000b).

In this example, both the upper opening and bottom of the object are described with petaloid curves. The mathematical equations of these two curves are

$$\begin{aligned} u = 0 \quad & x = r_1 \cos a_1 v + r_2 \cos a_2 v \\ & y = r_1 \sin a_1 v + r_2 \sin a_2 v \\ & z = h_0 \\ u = 1 \quad & x = r_3 \cos a_1 v + r_4 \cos a_2 v \\ & y = r_3 \sin a_1 v + r_4 \sin a_2 v \\ & z = 0 \end{aligned} \quad (3)$$

where $r_i, a_i (i = 1, \dots, 4)$ and h_0 are the design parameters.

Eq. (1) under the boundary conditions (3) can be solved by the separation of variables. For doing this we assume that the solutions of Eq. (1) have the form of

$$\begin{aligned} x &= q_{x1}(u) \cos a_1 v + q_{x2}(u) \cos a_2 v \\ y &= q_{y1}(u) \sin a_1 v + q_{y2}(u) \sin a_2 v \\ z &= c_0 + c_1 u \end{aligned} \quad (4)$$

Substituting Eq. (4) into Eq. (1) and making use of the boundary conditions (3), we obtain the following general solutions for the homogeneous equations of Eq. (1),

$$\begin{aligned}
x &= (c_{x1} e^{a_1 \sqrt{\frac{t_x}{s_x}} u} + c_{x2} e^{-a_1 \sqrt{\frac{t_x}{s_x}} u}) \cos a_1 v + (c_{x3} e^{a_2 \sqrt{\frac{t_x}{s_x}} u} + c_{x4} e^{-a_2 \sqrt{\frac{t_x}{s_x}} u}) \cos a_2 v \\
y &= (c_{y1} e^{a_1 \sqrt{\frac{t_y}{s_y}} u} + c_{y2} e^{-a_1 \sqrt{\frac{t_y}{s_y}} u}) \sin a_1 v + (c_{y3} e^{a_2 \sqrt{\frac{t_y}{s_y}} u} + c_{y4} e^{-a_2 \sqrt{\frac{t_y}{s_y}} u}) \sin a_2 v \\
z &= h_0(1 - u)
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
c_{i1} &= \frac{r_3 - r_1 e^{-a_1 \sqrt{\frac{t_i}{s_i}}}}{e^{a_1 \sqrt{\frac{t_i}{s_i}}} - e^{-a_1 \sqrt{\frac{t_i}{s_i}}}} \\
c_{i2} &= \frac{r_1 e^{a_1 \sqrt{\frac{t_i}{s_i}}} - r_3}{e^{a_1 \sqrt{\frac{t_i}{s_i}}} - e^{-a_1 \sqrt{\frac{t_i}{s_i}}}} \\
c_{i3} &= \frac{r_4 - r_2 e^{-a_2 \sqrt{\frac{t_i}{s_i}}}}{e^{a_2 \sqrt{\frac{t_i}{s_i}}} - e^{-a_2 \sqrt{\frac{t_i}{s_i}}}} \\
c_{i4} &= \frac{r_2 e^{a_2 \sqrt{\frac{t_i}{s_i}}} - r_4}{e^{a_2 \sqrt{\frac{t_i}{s_i}}} - e^{-a_2 \sqrt{\frac{t_i}{s_i}}}} \\
&(i = x, y)
\end{aligned} \tag{6}$$

The vector-valued parameters \mathbf{s} and \mathbf{t} in Eq. (1) will have significant influence on the shape of the surface. Taking $a_1 = 2\pi$, $a_2 = 16\pi$, $r_1 = 1$, $r_2 = 0.1$, $r_3 = 0.6$, $r_4 = 0.06$, and $h_0 = 3.6$, we obtain the surface in Figure 1a) when $s_x = s_y = 1$, $t_x = t_y = 0.5$ and obtain the surface in Figure 1b) when $s_x = s_y = 1$, $t_x = t_y = 0.1$.

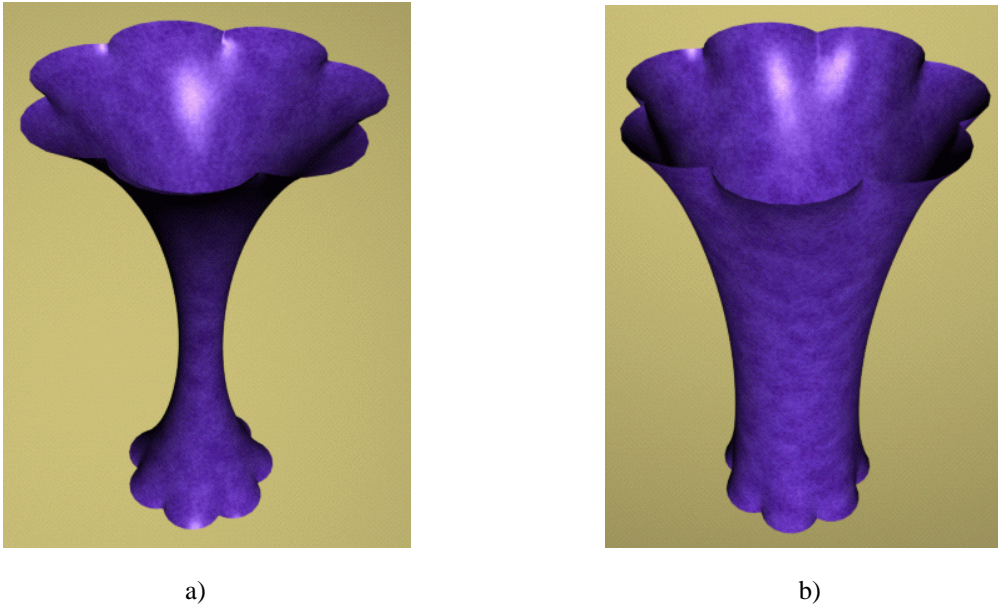


Figure 1. Surface generation with different vector-valued parameters in the second order PDE

The functional boundary conditions also have substantial effects on the shape of the surface. To demonstrate this, we only take the first term of x and y components in Eq. (3), i.e., setting $r_2 = r_4 = 0$. Keeping the other parameters unchanged, we obtain the shape in Figure 2a) when $s_x = s_y = 5$, $t_x = t_y = 0.2$, and the shape in Figure 2b) when $s_x = s_y = 1$, $t_x = t_y = 0.5$.

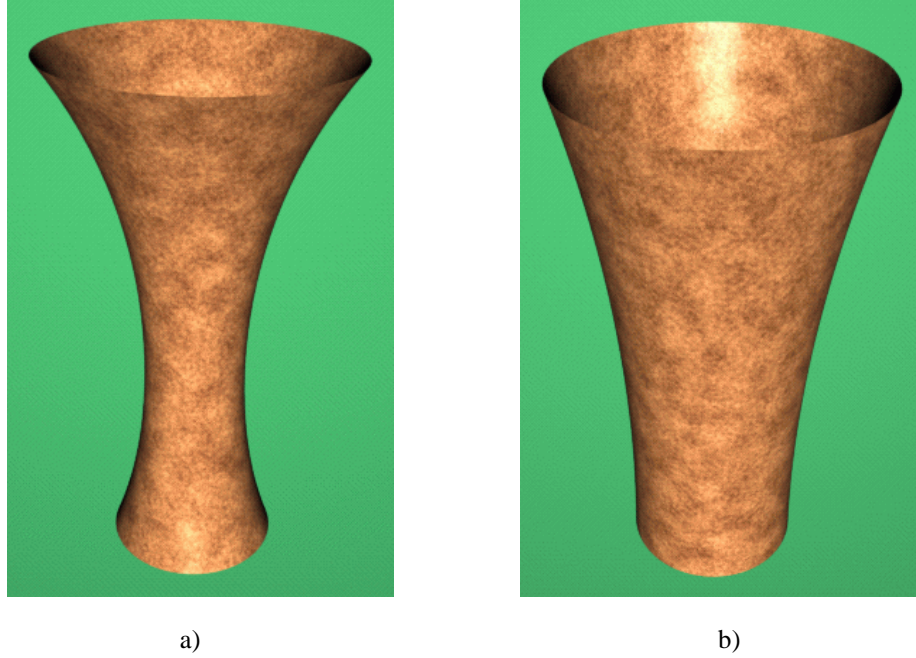


Figure 2 Surface generation with different functional boundary conditions in the second order PDE

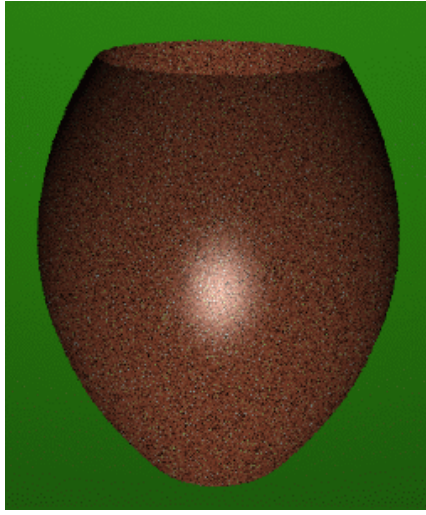
The force functions in Eq. (1) are another powerful tool for the manipulation of the surface geometry. In order to examine how these force functions affect the shape of the surface, let us consider the following force functions

$$\begin{aligned} p_x &= (p_0 + p_1 u) \cos v \\ p_y &= (p_0 + p_1 u) \sin v \\ p_z &= 0 \end{aligned} \quad (7)$$

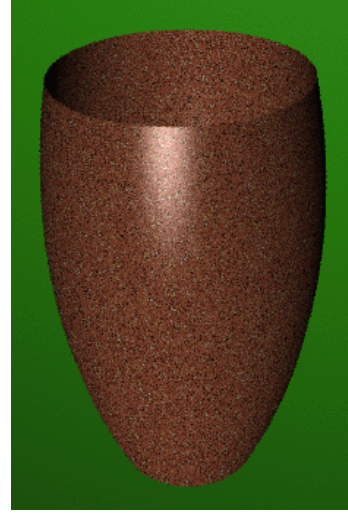
For simplicity, we still only take the first term of the x and y components of Eq. (3), i.e. setting $r_2 = r_4 = 0$. The introduction of the force functions leads to the following solutions

$$\begin{aligned} x &= (c_{x1} e^{a_1 \sqrt{\frac{t_x}{s_x}} u} + c_{x2} e^{-a_1 \sqrt{\frac{t_x}{s_x}} u} - \frac{p_0 + p_1 u}{t_x}) \cos a_1 v \\ y &= (c_{y1} e^{a_1 \sqrt{\frac{t_y}{s_y}} u} + c_{y2} e^{-a_1 \sqrt{\frac{t_y}{s_y}} u} - \frac{p_0 + p_1 u}{t_y}) \sin a_1 v \\ z &= h_0 (1 - u) \end{aligned} \quad (8)$$

With the above expressions, if we use the same geometric parameters as before and taking the vector-valued parameters to be $s_x = s_y = 5$, $t_x = t_y = 0.2$, we obtain the surface in Figure 3a) when $p_0 = -1$ and $p_1 = 0$; surface in Figure 3b) when $p_0 = -0.5$ and $p_1 = 0$; surface in Figure 3c) when $p_0 = 1.5$ and $p_1 = -4$; and surface in Figure 3d) when $p_0 = -3.1$ and $p_1 = 5$.



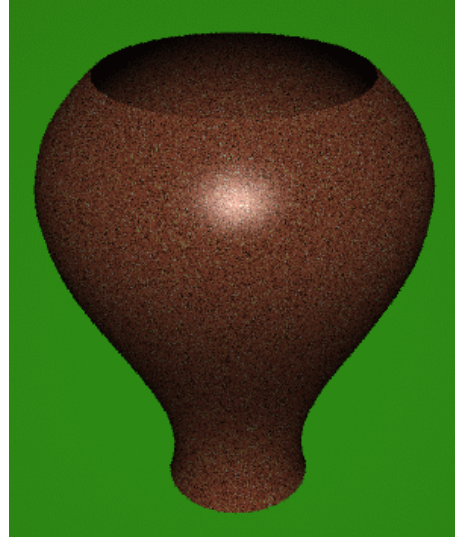
a)



b)



c)



d)

Figure 3 Surface generation with force functions in the second order PDE

It is clear that for each q_{ij} ($i = x, y; j = 1, 2$) in Eq. (4), there are only two unknown constants in comparison with four unknown constants in the solutions of the fourth order PDEs (You 1999b). Therefore, the computing time is halved or the speed is approximately doubled. In situations where speed is a crucial issue, this is no doubt an advantage. However, this second order PDE has a weak point, i.e. there is not enough flexibility to meet the tangential boundary conditions that again affects the shape of the surface. In addition, it can cope with fewer parameters than the PDEs of higher order. In order to create more complex shapes, the mixed order PDE is proposed in the following section.

SURFACE GENERATION WITH THE MIXED ORDER PDES

The inflexibility of the second order PDE proposed above can be improved by a mixed order PDE, so that the tangents at the boundaries can be accommodated. The x and y components of Eq. (1) are kept unchanged but its z component is replaced by a fourth order PDE. This mixed order PDEs can be written as

$$\begin{aligned}
s_x \frac{\partial^2 x}{\partial u^2} + t_x \frac{\partial^2 x}{\partial v^2} &= p_x(u, v) \\
s_y \frac{\partial^2 y}{\partial u^2} + t_y \frac{\partial^2 y}{\partial v^2} &= p_y(u, v) \\
s_z \frac{\partial^4 z}{\partial u^4} + t_z \frac{\partial^4 z}{\partial v^4} + w_z \frac{\partial^4 z}{\partial v^4} &= p_z(u, v)
\end{aligned} \tag{9}$$

Accordingly, the boundary conditions (2) can be modified as

$$\begin{aligned}
u = 0 \quad \mathbf{x} &= \mathbf{g}_1(u) \\
&\frac{\partial z}{\partial u} = g_3(u) \\
u = 1 \quad \mathbf{x} &= \mathbf{g}_2(u) \\
&\frac{\partial z}{\partial u} = g_4(u)
\end{aligned} \tag{10}$$

where $\mathbf{g}_1(u)$ and $\mathbf{g}_2(u)$ are vector-valued functions, and $g_3(u)$ and $g_4(u)$ are scalar functions.

In order to compare the second order PDEs with the mixed order PDEs for surface representation, we use the same boundary conditions of x and y components as we did with Eq. (3). But new boundary conditions are introduced for the z component. They are

$$\begin{aligned}
u = 0 \quad z &= h_0 \\
&\frac{\partial z}{\partial u} = h'_0 \\
u = 1 \quad z &= 0 \\
&\frac{\partial z}{\partial u} = 0
\end{aligned} \tag{11}$$

Without considering the force functions, the solutions of x and y components of Eq. (9) are the same as those of Eq. (5). However, the solutions of z component subject to boundary conditions (11) is now given by

$$z = h_0 + h'_0 u - (3h_0 + 2h'_0)u^2 + (2h_0 + h'_0)u^3 \tag{12}$$

Using the same parameters as the previous examples, we here examine how the tangential boundary conditions of z component affects the shape of the surface. When $s_x = s_y = 1$, $t_x = t_y = 0.5$, setting $h'_0 = -10$ creates the surface shown in Figure 4a) and $h'_0 = -0.01$ produces the surface in Figure 4b). When $s_x = s_y = 1$, $t_x = t_y = 0.1$, setting $h'_0 = -10$ generates the surface given in Figure 4c) and $h'_0 = -0.01$ produces the surface in Figure 4d). Comparing these pictures with those in Figure 1, it is evident that by using the mixed order PDEs (9), the tangential boundary conditions of z component enable more shapes to be created. This is done by choosing different values of the first derivative of the z component with respect to the u parameter.



a)



b)



c)



d)

Figure 4 Surface generation using the mixed order PDEs

SURFACE GENERATION WITH THE FOURTH ORDER PDES

From the above results, we have learned that by changing the vector-valued parameters in the partial differential equations and the boundary conditions, more variety of surfaces can be produced. Extending along this line, a fourth order PDE will be more powerful and flexible. Similar to those used in (You 1999a and Zhang 1999), the following fourth order PDE will provide more degrees of freedom not only to meet the tangential boundary conditions, but to allow the creation of much more variety of different shapes using the vector-valued parameters.

$$\mathbf{s} \frac{\partial^4 \mathbf{x}}{\partial u^4} + \mathbf{t} \frac{\partial^4 \mathbf{x}}{\partial u^2 \partial v^2} + \mathbf{w} \frac{\partial^4 \mathbf{x}}{\partial v^4} = \mathbf{p}(u, v) \quad (13)$$

where $\mathbf{w} = [w_x \ w_y \ w_z]^T$ is a vector-valued parameter.

Adding to Eq. (2), the new boundary conditions can be expressed as

$$\begin{aligned}
 u = 0 \quad \mathbf{x} &= \mathbf{g}_1(u) \\
 \frac{\partial \mathbf{x}}{\partial u} &= \mathbf{g}_2(u) \\
 u = 1 \quad \mathbf{x} &= \mathbf{g}_3(u) \\
 \frac{\partial \mathbf{x}}{\partial u} &= \mathbf{g}_4(u)
 \end{aligned} \tag{14}$$

where $\mathbf{g}_i(u)$ ($i = 1, 2, 3, 4$) are the vector functions.

In order to demonstrate the application of Eq. (13) in surface representation, especially its ability to produce different designs with the adjustable parameters, \mathbf{s} , \mathbf{t} and \mathbf{w} , let us consider the production of a family of pot-like objects. Both the upper opening and bottom profile of the surface are circles whose radii are r_1 and r_0 respectively. The height and the middle radius of the surface are h_1 and r_m , respectively. Based on the given geometric dimensions, we specify the boundary conditions as follows,

$$\begin{aligned}
 u = 0 \quad x &= r_0 \cos v & \frac{\partial x}{\partial u} &= \eta_1(4r_m - r_1 - 3r_0) \cos v \\
 y &= r_0 \sin v & \frac{\partial y}{\partial u} &= \eta_1(4r_m - r_1 - 3r_0) \sin v \\
 z &= 0 & \frac{\partial z}{\partial u} &= h'_0 \\
 u = 1 \quad x &= r_1 \cos v & \frac{\partial x}{\partial u} &= \eta_2(4r_m - r_1 - 3r_0) \cos v \\
 y &= r_1 \sin v & \frac{\partial y}{\partial u} &= \eta_2(4r_m - r_1 - 3r_0) \sin v \\
 z &= h_1 & \frac{\partial z}{\partial u} &= h'_1
 \end{aligned} \tag{15}$$

where h'_0 , h'_1 , η_1 , η_2 are the constants that are used to adjust the tangential boundary conditions.

The general solutions of Eq. (13) have two forms, which can be formulated as

$$\begin{aligned}
 x &= (c_1 e^{\xi_{x,1}u} + c_2 e^{\xi_{x,2}u} + c_3 e^{\xi_{x,3}u} + c_4 e^{\xi_{x,4}u}) \cos v \\
 y &= (c_5 e^{\xi_{y,1}u} + c_6 e^{\xi_{y,2}u} + c_7 e^{\xi_{y,3}u} + c_8 e^{\xi_{y,4}u}) \sin v \quad \text{for } s_x w_x < t_x^2 \text{ and } s_y w_y < t_y^2 \\
 z &= c_9 + c_{10}u + c_{11}u^2 + c_{12}u^3
 \end{aligned} \tag{16}$$

where

$$\begin{aligned}
 \xi_{i,1-4} &= \pm \sqrt{\frac{t_i}{2s_i} \left[1 \pm \sqrt{1 - \frac{4s_i w_i}{t_i^2}} \right]} \\
 (i &= x, y)
 \end{aligned}$$

and

$$\begin{aligned}
x &= \left[(c_1 + c_2 u) e^{\xi_{x,1} u} + (c_3 + c_4 u) e^{\xi_{x,2} u} \right] \cos v \\
y &= \left[(c_5 + c_6 u) e^{\xi_{y,1} u} + (c_7 + c_8 u) e^{\xi_{y,2} u} \right] \sin v \quad \text{for } s_x w_x = t_x^2 \text{ and } s_y w_y = t_y^2 \\
z &= c_9 + c_{10} u + c_{11} u^2 + c_{12} u^3
\end{aligned} \quad (17)$$

where

$$\begin{aligned}
\xi_{i,1-2} &= \pm \sqrt{\frac{t_i}{2s_i}} \\
(i &= x, y)
\end{aligned}$$

All the unknown constants c_i ($i = 1, 2, \dots, 12$) of Eqs. (16) and (17) can be determined by substituting these two equations into the boundary conditions (15).

In the above closed form solutions, by changing the functional and tangential boundary conditions as well as the vector-valued parameters, we are able to obtain a family of surfaces whose share fundamental features but are of different shapes. In Table 1, we list five combinations of the parameters where NOC stands for the case number and NOF for the figure number in Figure 5.

Table 1 Different combinations of boundary conditions and vector-valued parameters

NOC	NOF	r_0	r_1	r_m	h_1	η_1	η_2	h'_0	h'_1	s_x	t_x	w_x	s_y	t_y	w_y
1	a	0.4	0.3	0.8	2	0.5	0.5	6	6	1	3	1	1	3	1
2	b	0.2	0.8	0.7	2	0.1	0.1	6	6	1	16	60	1	16	60
3	c	0.4	0.3	0.2	2	2.5	2.5	6	6	1	6	1	1	6	1
4	d	0.4	0.3	0.8	3	0.3	0.3	1.5	1.5	1	15	1	1	15	1
5	e	0.25	0.5	0.3	3	0.5	30	1.5	9	1	3	2	1	3	2



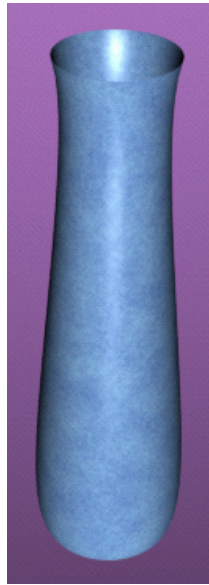
a)



b)



c)



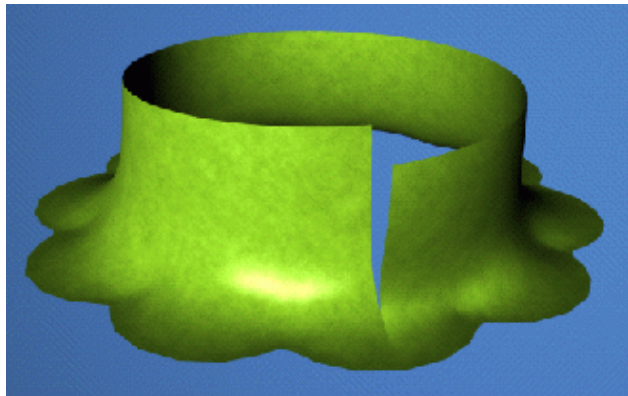
d)



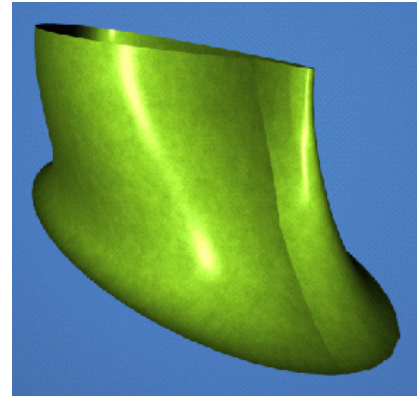
e)

Figure 5 Surface generation using the fourth order PDEs

What is worth pointing out is that the powerfulness and the flexibility of this fourth order PDE Eq.(13) do not just stop here. As a matter of fact, it is able to produce free-form surfaces of much greater complexity. Illustrated in figure 6 are two such examples created using the same PDE



a)



b)

Figure 6 Surfaces of more complexity

DISCUSSION AND CONCLUSION

Partial differential equations are powerful tools for the generation of free-form surfaces. Compared with the conventional surface modelling approaches, the PDE based approach is more convenient for the creation of families of surfaces by adjusting the parameters in both the PDEs and the boundary condition expressions.

To allow optimum determination of the geometric quality and computational speed in surface representation, in this paper, we have put forward three types of PDEs. The forms of the PDEs, the functional and tangential boundary conditions and the closed form solutions of these partial differential equations have been formulated. The performance and the ability of shape control for the generated surfaces are discussed along with the illustration of a number of surface design examples.

It is observed that the second order PDEs have the highest computational efficiency amongst the three forms of PDEs. This is because this form contains the least number of unknown constants to be determined. Due to exactly the same reason, this form is the least powerful in terms of the variety of the surfaces it can generate. Without sacrificing perceptible computational performance, the second order PDEs can be improved by introducing the fourth order PDEs for one of the components leading to the mixed order PDEs. This new collocation allows the tangential boundary conditions to be fulfilled for the components concerned. As a consequence, more complex boundary conditions can be met. The fourth order partial differential equations, despite their lower efficiency than the second and mixed order PDEs, have much more degrees of freedom, and hence are able to generate family of surfaces with sophisticated geometric features. These degrees of freedom are realised through the use of the vector-valued parameters, the functional boundary conditions and the tangential boundary conditions.

These three forms of PDEs should be applied in different situations depending on the actual applications. If real-time performance is a crucial issue, such as in virtual reality applications, the second and mixed order PDEs should be considered first. If the shape requirement and modelling accuracy become a dominating factor, the fourth order PDEs of three vector-valued parameters should be given a priority.

Acknowledgement: The authors would like to gratefully acknowledge Professor P Comninou for his valuable discussions.

REFERENCES

1. Bloor, M. I. G., and Wilson, M. J., 1990, Using partial differential equations to generate free-form surfaces, *Computer-Aided Design*, 22(4), 202-212.
2. Bloor, M. I. G., and Wilson, M. J., 1994, Local control of surfaces generated using partial differential equations, *Computers & Graphics*, 18(2), 161-169.
3. Bloor, M. I. G., and Wilson, M. J., 1995, Complex PDE surface generation for analysis and manufacture, *Computing*, 10, 61-77.
4. Bloor, M. I. G., and Wilson, M. J., 1996, Spectral approximations to PDE surfaces, *Computer-Aided Design*, 28(2), 145-152.
5. Brown, J. M., Bloor, M. I. G., Bloor, M. S., and Wilson, M. J., 1990, Generation and modification of non-uniform B-spline surface approximations to PDE surfaces using the finite element method, *Adv. Des. Automation*, 265-272.
6. Lowe, T. W., Bloor, M. I. G., and Wilson, M. J., 1990, Functionality in surface design, *Adv. Des. Automation*, 43-50.
7. You, L. H., Zhang, J. J., and Comninou, P., 1999a, Cloth deformation modelling using a plate bending model, *The 7th International Conference in Central Europe on Computer Graphics, Visualisation and Interactive Digital Media*, 485-491.
8. You, L., and Zhang, J. J., 1999b, Blending surface generation with a fourth order partial differential equation, *The Sixth International Conference on Computer-Aided Design and Computer Graphics (CAD/Graphics'99)*, Shanghai, China, 1-5 December.
9. You, L. H., and Zhang, J. J., 2000a, Free form surface generation using analytical solution to a fourth order partial differential equation, *The Fifth World Conference on Integrated Design and Process Technology*, Texas, USA, June.
10. You, L. H., Zhang, J. J. and Comninou, P., 2000b, A volumetric deformable muscle model for computer animation using weighted residual method, *Computer Methods in Applied Mechanics and Engineering*, accepted for publication and in press.
11. Zhang, J. J., You, L. H., and Comninou, P., 1999, Computer simulation of flexible fabrics, *EUROGRAPHICS*, UK Chapter, 27-35.

12. Zhang, J. J. and You, L. H., 2000, Surface blending using a power series solution to fourth order partial differential equations, ACM Transactions on Graphics, under review.