Lecture 3 Cartesian

The Greek Alphabet

$\mathrm{A}\alpha$ Alpha	$\mathrm{B}\beta$ Beta
$\Gamma\gamma$ Gamma	$\Delta\delta$ Delta
$\mathrm{E}\epsilon\varepsilon$ Epsilon	$\mathrm{Z}\zeta$ Zeta
$\mathrm{H}\eta$ Eta	$\Theta\theta$ Theta
$\mathrm{I}\iota$ Iota	$\mathrm{K}\kappa$ Kappa
$\Lambda\lambda$ Lambda	$\mathrm{M}\mu$ Mu
$\mathrm{N}\nu$ Nu	$\Xi\xi$ Xi
$\mathrm{O}o$ Omicron	$\Pi\pi$ Pi
$\mathrm{P}\rho$ Rho	$\Sigma\sigma\varsigma$ Sigma
$\mathrm{T}\tau$ Tau	$\Upsilon\upsilon$ Upsilon
$\Phi\phi\varphi$ Phi	$\mathrm{X}\chi$ Chi
$\Psi\psi$ Psi	$\Omega\omega$ Omega

Symbols

- Mathematicians use all sorts of symbols to substitute for natural language expressions.
- Here are some examples

$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\approx$	approximately equal
$\equiv$	equivalent to
$\neq$	not equal to

Fibonacci Numbers

- Fibonacci sequence is the sequence of integers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

- where each number is the sum of the previous two
- It can be defined recursively as

$$F_n := F(n) := \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F(n-1) + F(n-2) & \text{if } n > 1. \end{cases}$$

The Golden Ratio

- The golden ratio is an irrational number, approximately 1.61803..., that possesses many interesting properties.
- It is usually represented by the Greek letter lower case Phi φ sometimes it is also written with the uppercase version ϕ
- Shapes defined by the golden ratio have long been considered aesthetically pleasing in Western cultures
- They are said to reflecting nature's balance between symmetry and asymmetry
- It can also be found in the Music of Mozart, Bach, Bartók, Debussy, Schubert...
- The golden ratio is also referred to as the golden mean, golden section, golden number, divine proportion or sectio divina

Golden Ratio

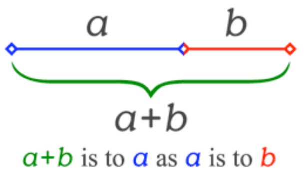
- As pointed out by Johannes Kepler, the ratio of consecutive Fibonacci numbers

$$\frac{F(n+1)}{F(n)}$$

- converges to the golden section φ defined as the positive solution of the equation

$$\frac{x}{1} = \frac{1}{x-1} \text{ or equivalently } x = 1 + \frac{1}{x}$$

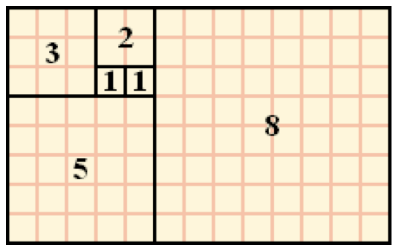
Golden Ratio Line Section



- The golden ratio φ represented as a line divided into two segments a and b, such that the entire line is to the longer a segment as the a segment is to the shorter b segment.

Golden Ratio Rectangle

- The above image shows the Fibonacci Sequence and the relationship to the Golden Rectangle



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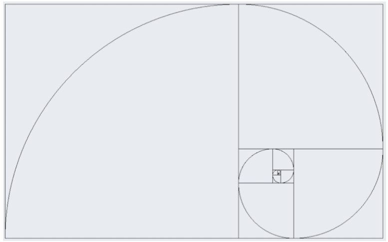
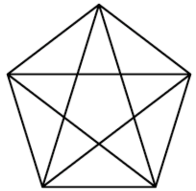


Golden Ratio Definition

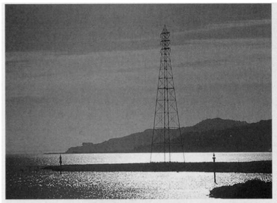


- Mathematically the Golden Ratio can be defined by the following equation

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618033988\ldots$$

- The number frequently turns up in Geometry and in particular in figures involving pentagonal symmetry

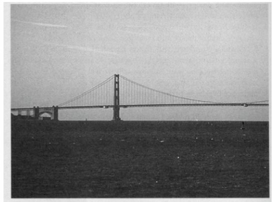
Golden Ratio in Film



8.20 Golden Section Applied: Horizontal Division
It is especially effective to have a single vertical element divide a clean horizontal graphic vector at the golden section.

- Sergie Eisenstein directed the classic silent film of 1925 The Battleship Potemkin
- He divided the film up using golden section points to start important scenes in the film, measuring these by length on the celluloid film.
- The same proportions are also commonly used to divide screen space when framing a shot (as seen in the images above)
- It is also common to see the ratio in title sequence design

Golden Ratio in Film



8.21 Golden Section Applied: Vertical Division
You can also use horizontal graphic vectors (such as the horizon) to divide the screen vertically at the golden section.

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Law of Indices

- The Law of Indices can be expressed as

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(a^m)^n = a^{mn}$$

- Examples

$$2^3 \times 2^2 = 8 \times 4 = 32 = 2^5$$
$$2^4 \div 2^2 = 16 \div 4 = 4 = 2^2$$
$$(2^2)^3 = 64 = 2^6$$

Law of Indices

- From the previous examples, it is evident that

$$a^0 = 1$$
$$a^{-p} = \frac{1}{a^p}$$
$$a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

Logarithms

- Two people are associated with logarithms:
- John Napier (1550-1617) and Joost Bürgi (1552-1632).
- Logarithms exploit the addition and subtraction of indices and are always associated with a base
- For Example, if

$$a^x = n$$
$$\log_a n = x$$
$$\text{Where a is the base.}$$

Logarithms

$$10^2 = 100$$
$$\log_{10} 100 = 2$$

- It can be said "10 has been raised to the power 2 to equal 100"
- The log operation finds the power of the base for a given number

Logarithms

- Multiplication's can be translated into an addition using logs

$$36 \times 24 = 864$$
$$\log_{10} 36 + \log_{10} 24 = \log_{10} 864$$
$$1.5563025007 + 1.38021124171 = 2.963651374248$$

- The two bases used in calculators and computer software are 10 and 2.718281846..., the second value is know as the transcendental number e
- Logs to the base 10 are written as ***log***
- Logs to the base e are written as ***ln***

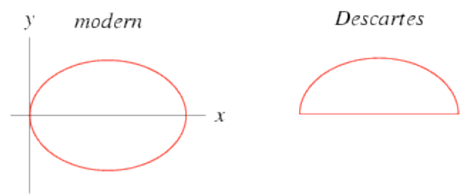
Logarithms

$$\log(ab) = \log a + \log b$$
$$\log(\tfrac{a}{b}) = \log a - \log b$$
$$\log(a^n) = n \log a$$
$$\log(\sqrt[n]{a}) = \tfrac{1}{n} \log a$$

Cartesian Co-ordinates

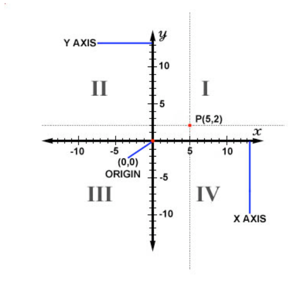
- In 1636 Fermat (1601-1665) was working on a treatise titled "Ad locus planos et solidos isagoge" which outlined what we now call analytic geometry.
- Fermat never published his treatise, but shared his ideas with other mathematicians such as Blaise Pascal (1623-1662).
- In 1637 René Descartes (1596-1650) devised his own system of analytic geometry and published his results in the prestigious journal Géométrie.
- Ever since this publication Descartes has been associated with the xy-plane, which is why it is called the Cartesian plane.
- If Fermat had been more efficient with publishing his research results, the xy-plane could have been called the Fermatian plane!

Cartesian Co-ordinates



- In René Descartes' original treatise the axes were omitted, and only positive values of the x- and the y- co-ordinates were considered, since they were defined as distances between points.
- For an ellipse this meant that, instead of the full picture which we would plot nowadays (left figure), Descartes drew only the upper half (right figure).

Cartesian Co-ordinates



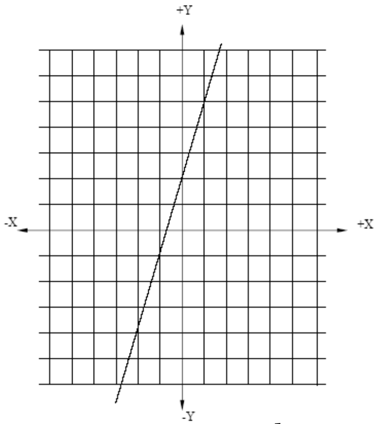
2 DIMENSIONAL CARTESIAN COORDINATE SYSTEM

•The modern Cartesian co-ordinate system in two dimensions (also called a rectangular co-ordinate system) is commonly defined by two axes, at right angles to each other, forming a plane (an xy-plane).
•The horizontal axis is labelled x, and the vertical axis is labeled y.
•In a three dimensional co-ordinate system, another axis, normally labeled z, is added, providing a sense of a third dimension of space measurement.
•The axes are commonly defined as mutually orthogonal to each other (each at a right angle to the other).
•All the points in a Cartesian co-ordinate system taken together form a so-called Cartesian plane.
•The point of intersection, where the axes meet, is called the origin normally labelled O.
•With the origin labelled O, we can name the x axis O_x and the y axis O_y.
•The x and y axes define a plane that can be referred to as the xy plane. Given each axis, choose a unit length, and mark off each unit along the axis, forming a grid.

Cartesian Co-ordinates

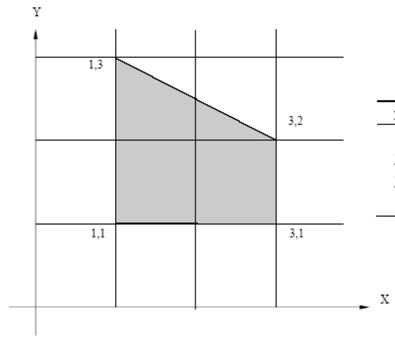
- To specify a particular point on a two dimensional co ordinate system, you indicate the x unit first (abscissa), followed by the y unit (ordinate) in the form (x,y), an ordered pair.
- In three dimensions, a third z unit (applicate) is added, (x,y,z).

Cartesian Co-ordinates



The equation $y = 3x + 2$ using the x-y Cartesian plane

Geometric Shapes



x	y
1	1
3	1
3	2
1	3

- A Simple polygon created with the four vertices shown in the table

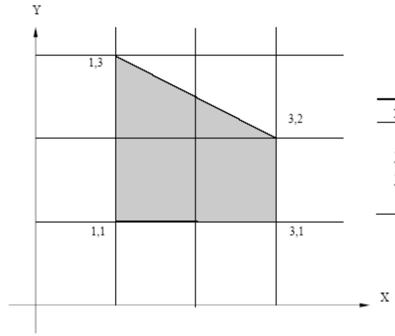
Areas of Shapes

- The area of a polygonal shape is computed by using the vertices by the following

x	y
x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3

$$\frac{1}{2}[(x_0y_1 - x_1y_0) + (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_0 - x_0y_3)]$$

Geometric Shapes



x	y
1	1
3	1
3	2
1	3

$$\frac{1}{2}[(1\times 1 - 3\times 1) + (3\times 2 - 3\times 1) + (3\times 3 - 1\times 2) + (1\times 1 - 3\times 1)]$$

$$\frac{1}{2}[-2 + 3 + 7 - 2] = 3$$

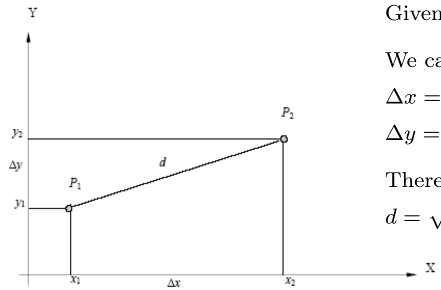
Reverse Order

x	y
1	1
1	3
3	2
3	1

$$\frac{1}{2}[(1\times 3 - 1\times 1) + (1\times 2 - 3\times 3) + (3\times 1 - 3\times 2) + (3\times 1 - 1\times 1)]$$

$$\frac{1}{2}[2 - 7 - 3 + 2] = -3$$

Theorem of Pythagorus in 2D



Given two arbitrary points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

We can calculate the distance between the two points

$\Delta x = x_2 - x_1$

$\Delta y = y_2 - y_1$

Therefore, the distance d between $P_1$and P_2 is given by :

$d = \sqrt{\Delta x^2 + \Delta y^2}$

References

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- http://en.wikipedia.org/wiki/Cartesian_coordinate_system