# *Mathematics for Computer Graphics*

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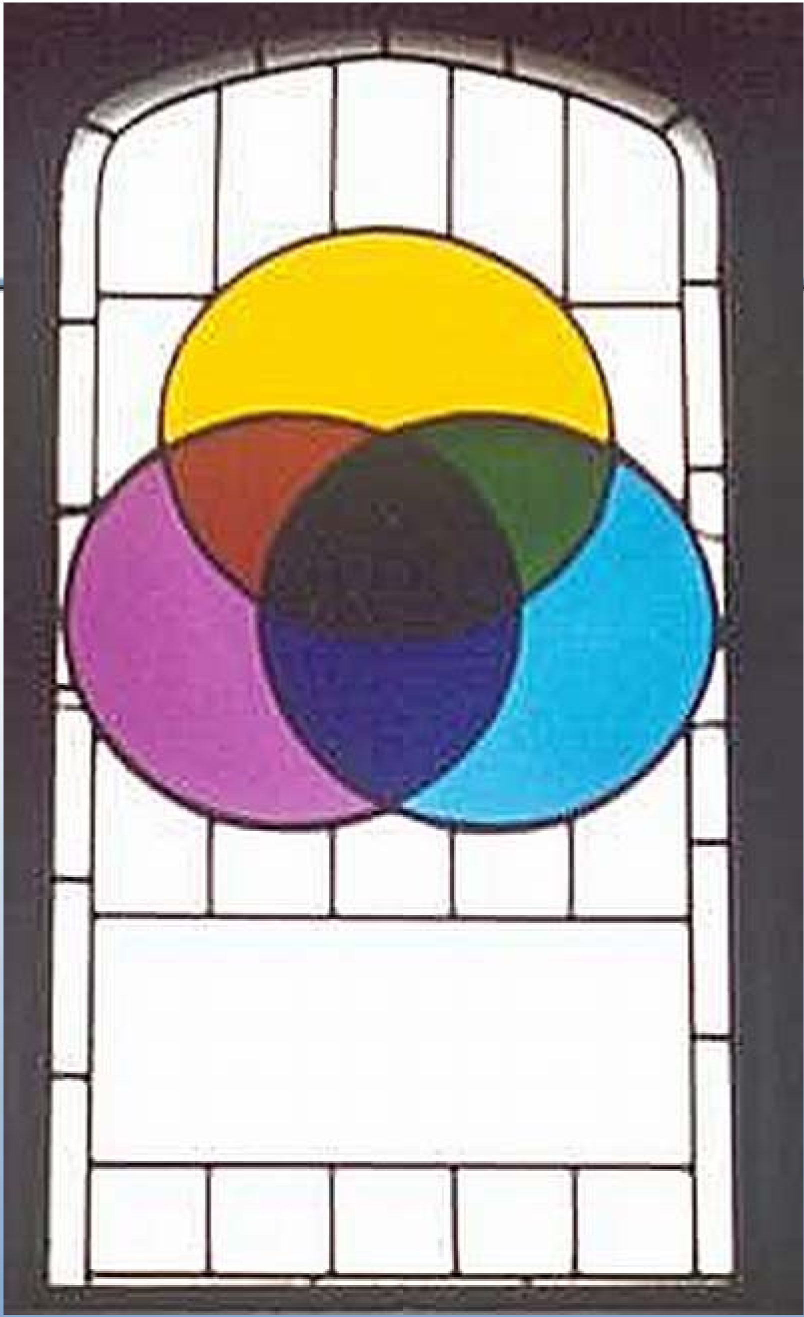
# *Sets and Functions*



The stained glass in Caius Hall at Cambridge University commemorating John Venn

# Contents

- Notions for sets
- Venn diagrams
- Basic set operations
- Notions of functions
- Properties of functions

# Set theory

- Creation of one mathematician: Georg Cantor (1845-1918), born in Russia to a Danish father and a Russian mother and spent most of his life in Germany
- Great importance to the modern formulation of many topics of continuous and discrete mathematics

Georg Cantor
1845-1918

# Notion of a Set

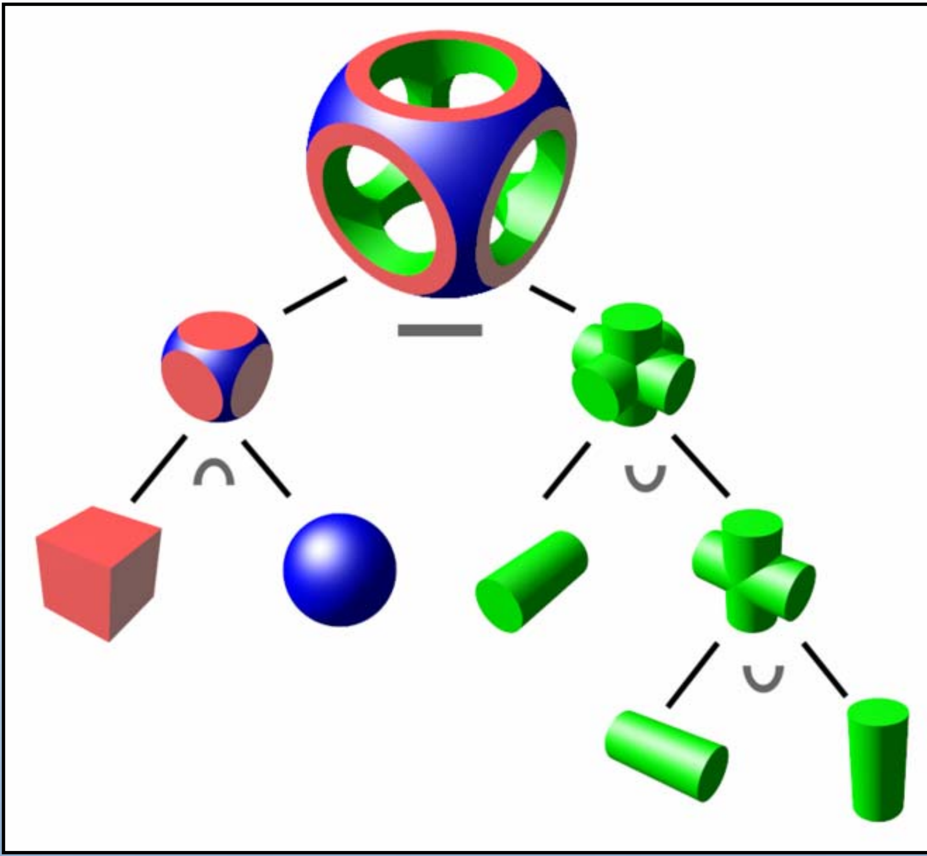
- A *set* is a type of structure, representing an *unordered* collection of zero or more *distinct* objects (elements).
- Set theory deals with operations between, relations among, and statements about sets.
- *All* of mathematics can be defined in terms of some form of set theory.
- Sets are extensively used in computer software systems.

# Intuition behind Sets

- The elements of a set can be anything: numbers, people, letters of the alphabet, other sets, and so on.
- Almost anything you can do with individual objects, you can also do with sets of objects:
  - refer to them, compare them, combine them, …
- You can also do some things to a set that you probably cannot do to an individual:
  - check whether one set is contained in another
  - determine how many elements it has
  - quantify over its elements (using it for $\exists, \forall$)

# Why do we need sets?

- Universal language for explaining mathematical concepts
- Practical geometric modelling language - Constructive Solid Geometry (CSG)
- $\cup$, $\cap$, **-** : set operations



# Basic Notations for Sets

- Sets are conventionally denoted with capital letters *S*, *T*, *U*, …
- *We may define a particular set in two distinct ways:*
  - *A = {2 ,3, 6 ,8}      tabular form of the set.*
  - *B = {x | x is an odd integer} or*
    *B = {x : x is an odd integer}.*
    *Here the symbols " | " and " : " are read as "where".*
- *A more general form (a set-builder form):*
  *S = {x | P(x )} denotes the set S of all the entities (objects) x for which the condition (proposition) P( x ) holds true*

# Set Membership

If an object *x* is a member of a set ***A*** , then we denote this relationship as: $x \in A$ which reads " x belongs to A ", " x is a member of A " or " x is in A ".

If an object x is not a member of a set A , then we denote this relationship as: $x \notin A$ which reads " x does not belong to A ", " x is not a member of A " or " x is not in A ".

The symbol " $\in$ " was introduced by the Italian mathematician Giuseppe Peano in 1888.

# Finite and Infinite Sets

- We say that a set is finite if it consists of a specific number of different elements. Otherwise, we say that the *set is infinite*. For instance:
  - If ***D*** is the set of the days of the week, then ***D*** is a *finite set.*
  - If ***O*** = {1, 3, 5, 7, …} , then ***O*** is an *infinite set*.
- If a set ***S*** has *n* elements (where *n* is non-negative integer), then we say that S has ***cardinality n*** .

# Basic Properties of Sets

- Sets are inherently *unordered*:
  - No matter what objects a, b, and c denote,
    {a, b, c} = {a, c, b} = {b, a, c} = …
- Multiple listings make no difference:
  - {a, a, c, c, c, c}={a,c}.

# Basic properties of sets

- A set $A$ is said to be equal to a set $B$ , if both sets have the same members. We denote this equality as $A = B$
- If the two sets are not equal, then we write $A \neq B$
  - *If* $A = \{1, 2, 3, 4\}$ *and* $B = \{3, 1, 4, 2\}$ *, then* $A = B$
  - *If* $C = \{5, 6, 5, 7\}$ *and* $D = \{7, 5, 7, 6\}$ *, then* $C = D$

# The Empty Set

- A set that contains no elements is called a null set or an empty set and is denoted by the symbol " $\varnothing$ ".
  - *If A is the set of all people in the world who are older than 200 years, then A is the empty set, i.e. $A = \varnothing$.*
- The empty set is the unique set that can be defined as

$$\varnothing = \{\} = \{x \mid x \neq x\} = \ldots = \{x \mid \textbf{False}\}$$

# Subsets and Supersets

- If every element of a set ***A*** is also an element of a set ***B***,
  then set **A** is called ***a subset of set*** **B**. is denoted as ***A*** $\subseteq$ ***B and*** reads " ***A*** *is a subset of* **B** " or " ***A is contained in*** **B**".
  - If $C = \{1,3,5\}$ and $D = \{5,4,3,2,1\}$, then $C \subseteq D$.
  - If $E = \{2,4,6\}$ and $F = \{6,4,2\}$, then $E \subseteq F$.
- If set $A$ is a subset of set $B$ ($A \subseteq B$), then we can also denote this as ***B*** $\supseteq$ ***A***, which reads " *B is a superset of A* " or " ***B contains A*** "
- The null set $\varnothing$ is a subset of every set

# Proper (Strict) Subsets & Supersets

- $S \subset T$ ("$S$ is a proper subset of $T$") means that $S \subseteq T$ but $T \not\subseteq S$.

Example: $\{1,2\} \subset \{1,2,3\}$

We have $\{1,2,3\} \subseteq \{1,2,3\}$,

but not $\{1,2,3\} \subset \{1,2,3\}$

# Sets Are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.
- Example: let $S=\{x \mid x \subseteq \{1,2,3\}\}$
  then $S=\{\emptyset,$
  $\{1\}, \{2\}, \{3\},$
  $\{1,2\}, \{1,3\}, \{2,3\},$
  $\{1,2,3\}\}$
- Note that $1 \neq \{1\} \neq \{\{1\}\}$

# Cardinality and Finiteness

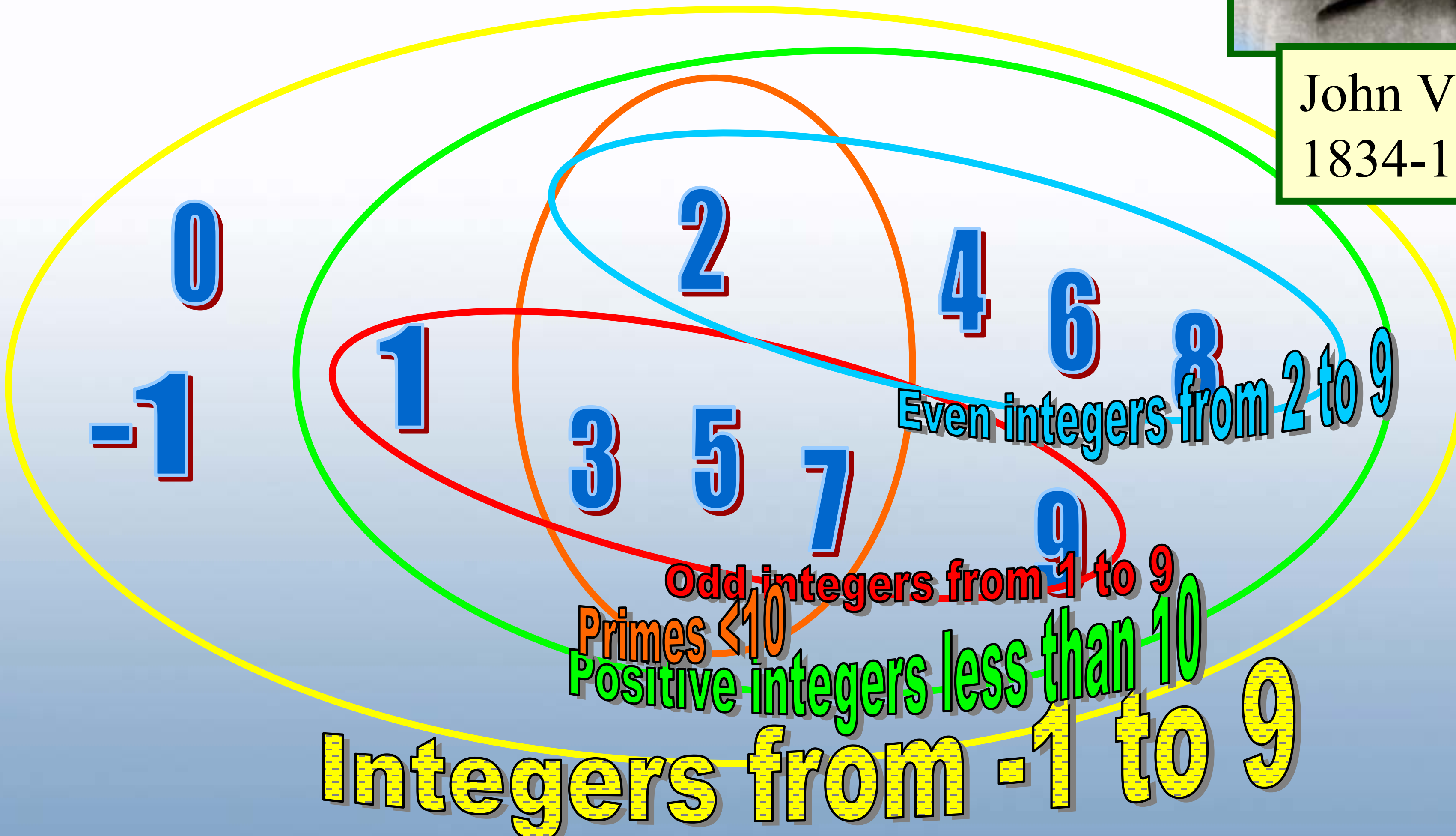
- $|S|$ (read "the *cardinality* of *S*") is a measure of how many different elements *S* has.
- *E.g.*, $|\emptyset|=0$, $|\{1,2,3\}| = 3$, $|\{a,b\}| = 2$, $|\{\{1,2,3\},\{4,5\}\}| = $ 2
- If $|S| \in \mathbf{N}$, then we say *S* is *finite*. Otherwise, we say *S* is *infinite*.

# Power Set

- The *power set* P(*S*) of a set *S* is the set of all subsets of *S*. $P(S) :\equiv \{x \mid x \subseteq S\}$.
- Example: P({a,b}) = {∅, {a}, {b}, {a,b}}.
- Sometimes P(*S*) is written $\mathbf{2}^S$, because $|P(S)| = 2^{|S|}$.
- It turns out $\forall S: |P(S)| > |S|$, *e.g.* $|P(\mathbf{N})| > |\mathbf{N}|$. *There are different sizes of infinite sets.*

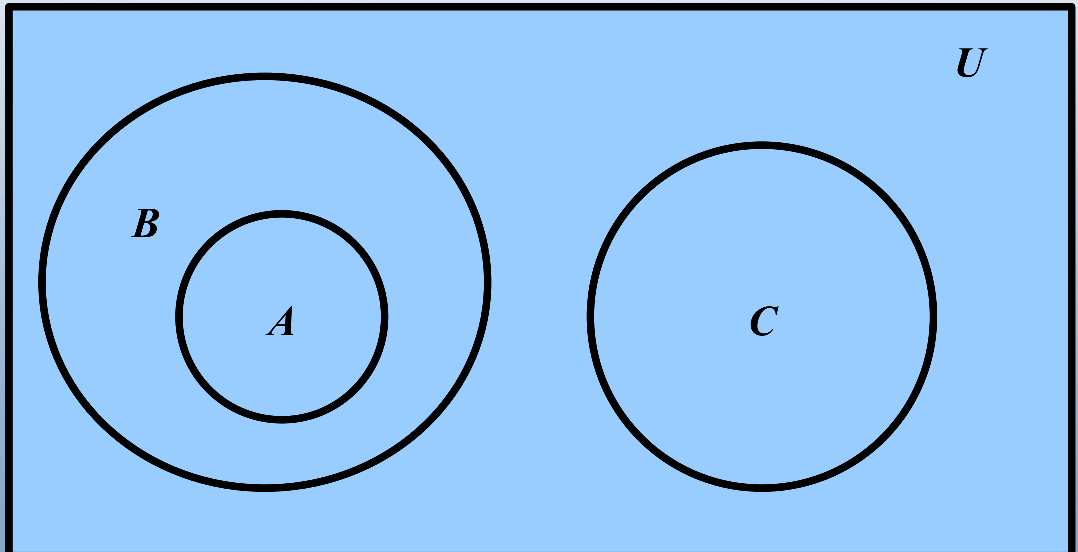
# Venn Diagrams

John Ven
1834-192

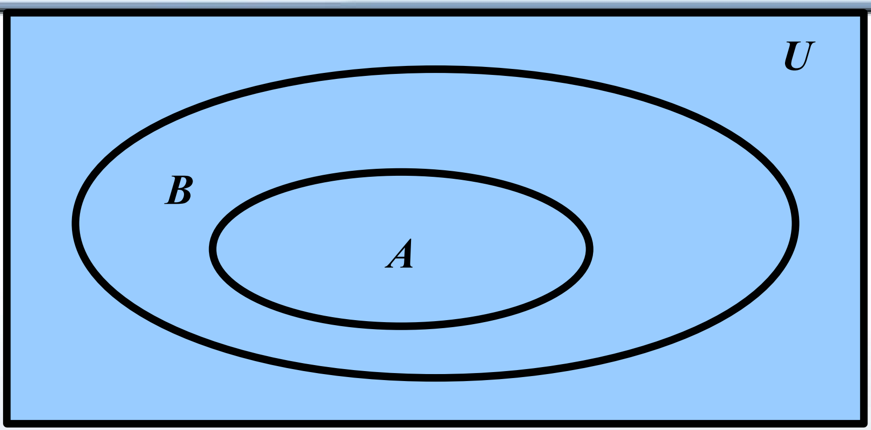


- In any application of set theory, all the sets under investigation are subsets of a fixed set. We call this set the ***universal set U*** or the ***universe of discourse U.***

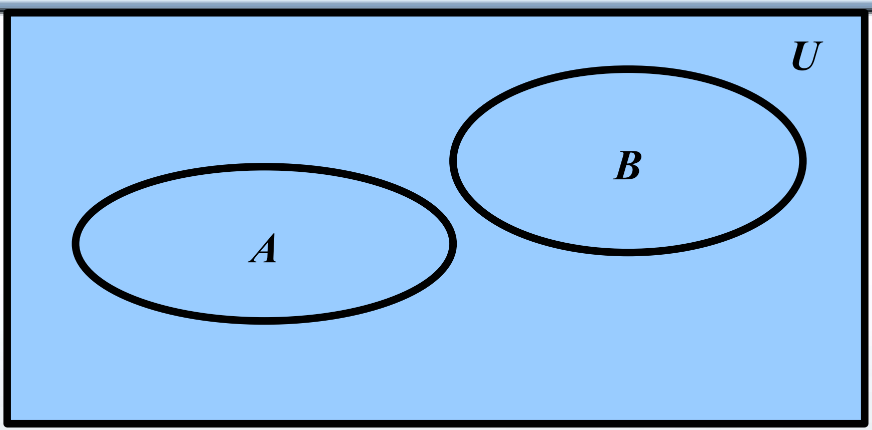
Example: The universal set U represents all animals, C represents the set of all camels, B represents the set of all birds and A represents the set of all albatrosses, then the Venn diagram represents the relationship of these sets.



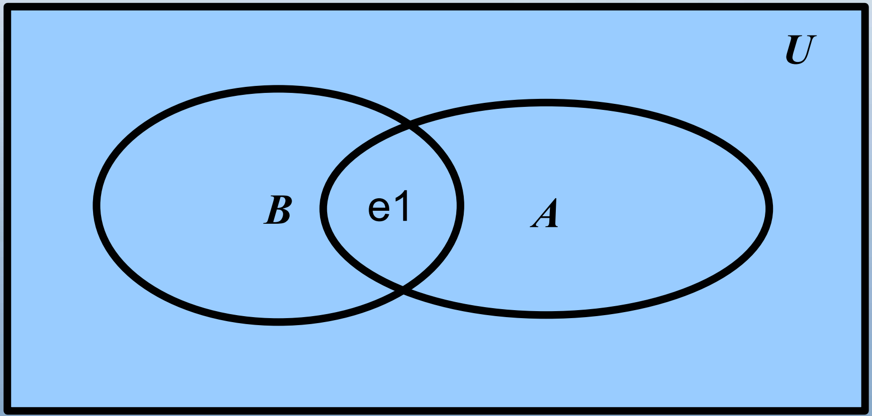
# Venn Diagrams




The Venn diagram of A ⊆ B




The Venn diagram of two disjoint sets.




The Venn diagram of two sets that share some common elements.

# Contents

- Notions for sets
- Venn diagrams
- Basic set operations
- Notions of functions
- Properties of functions

# Basic Set Operations: Union

The ***union*** of sets ***A*** and ***B*** is the set of elements that belong to set ***A*** or to set ***B*** or to both sets. We denote the union of sets ***A*** and B by ***A*** ∪ ***B*** , which reads " ***A*** *union* ***B*** ".

– $A \cup B = \{x \mid x \in A \lor x \in B\}$

Example: if ***A*** = {a , b , c , d } and B = {c , d , e , f } then ***A*** ∪ ***B*** = {a , b , c , d, e , f }.

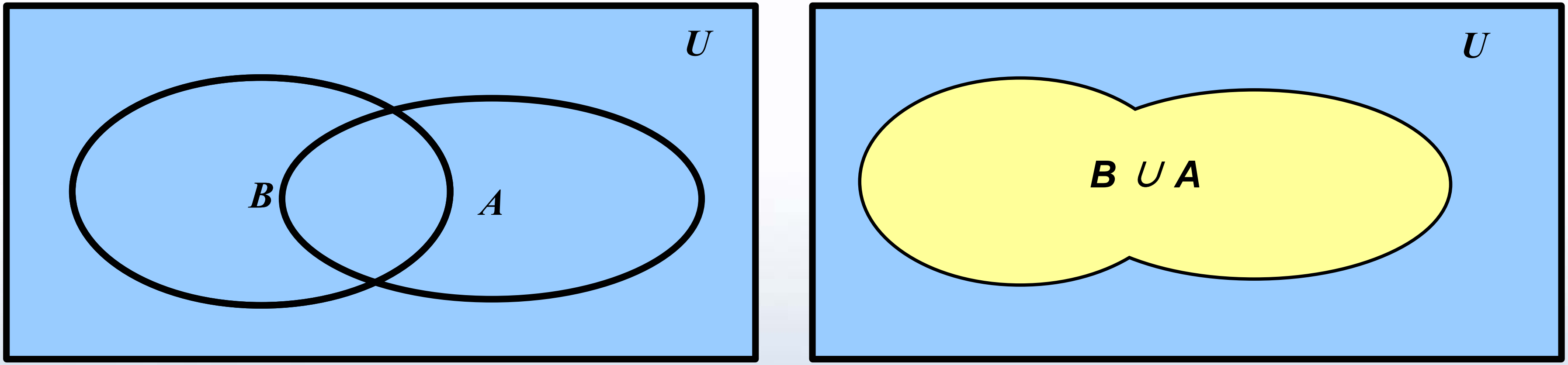
– The union operation is commutative

$$A \cup B = B \cup A$$

– Both sets are subsets of their union

$$A \subseteq (A \cup B) \text{ and } B \subseteq (A \cup B).$$

# Basic Set Operations: Union

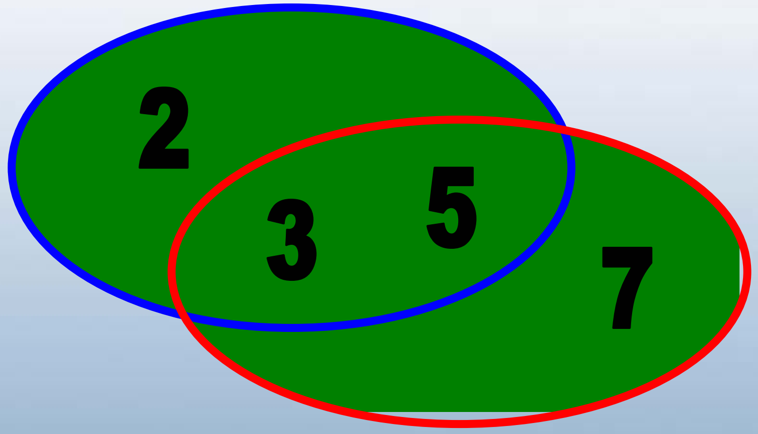



Venn diagram for the ***union*** of sets ***B*** and ***A***

$$B \cup A$$

# Union Examples

- {a,b,c} $\cup$ {2,3} = {a,b,c,2,3}
- {2,3,5} $\cup$ {3,5,7} = {2,3,5,3,5,7} = {2,3,5,7}



# Intersection operation

The ***intersection*** of sets ***A*** and ***B*** is the set of elements that are common to both sets. We denote the intersection of sets ***A*** and ***B*** by ***A*** ∩ ***B*** , which reads " ***A*** *intersection* ***B*** ":

- $\boldsymbol{A} \cap \boldsymbol{B} = \{x \mid x \in \boldsymbol{A} \land x \in \boldsymbol{B}\}$

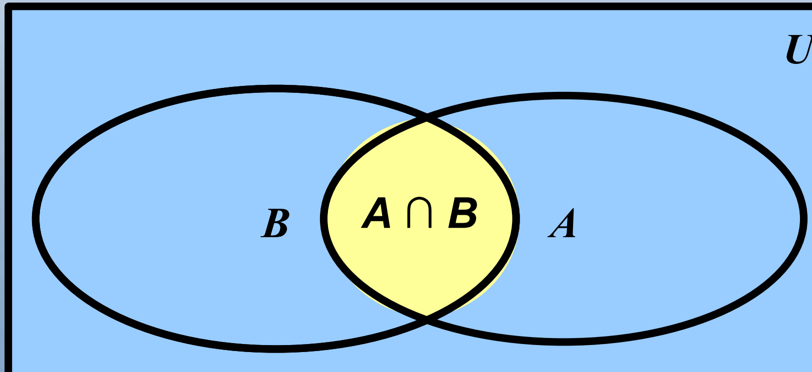
If ***A*** = {a , b , c , d } and B = {c , d , e , f } , then ***A***∩ ***B*** = {c , d }.

- The *intersection* is commutative

$$A \cap B = B \cap A.$$

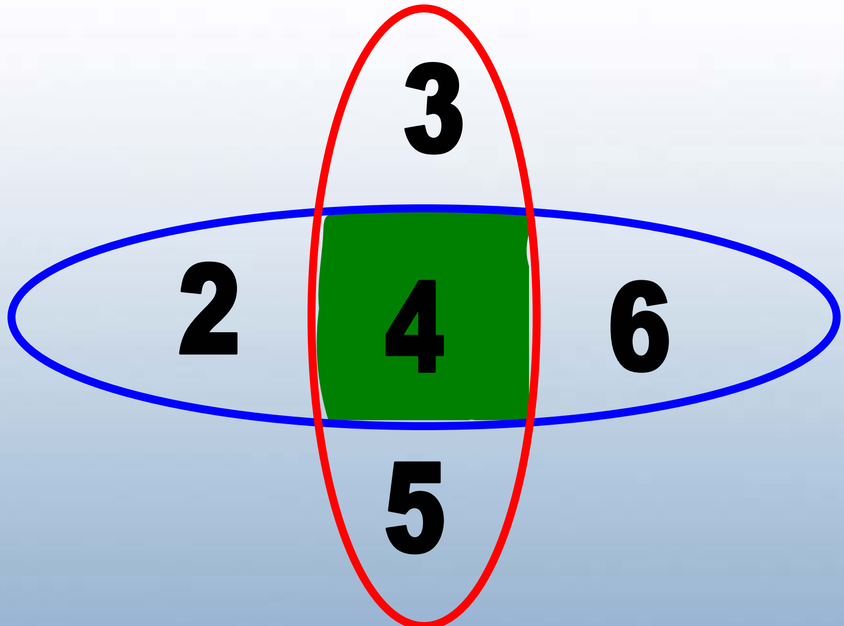
- the *intersection* of two sets is subset of both sets

$(A \cap B) \subseteq A$ and $(A \cap B) \subseteq B$ .



# Intersection Examples

- $\{a,b,c\} \cap \{2,3\} = \underline{\ \emptyset\ }$
- $\{2,4,6\} \cap \{3,4,5\} = \underline{\ \{4\}\ }$

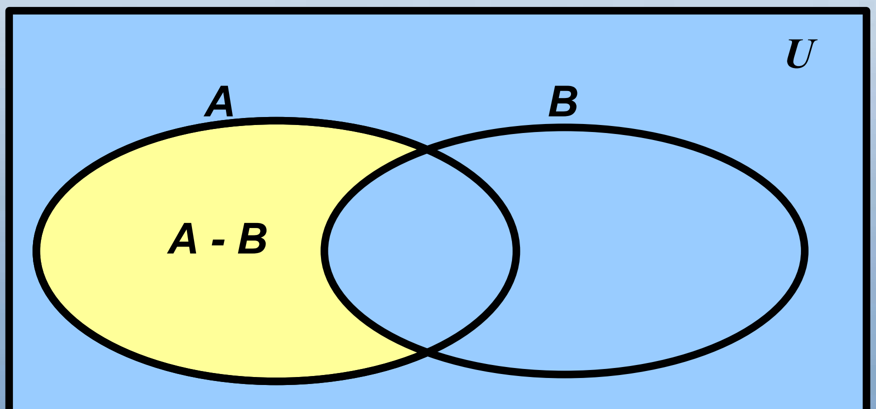


# Difference operation

- The ***difference*** of sets ***A*** and ***B*** (subtraction of ***B*** from ***A***) is th set of elements that belong to set A and do not belong to set B. We denote the *difference* of sets ***A*** and ***B*** by ***A - B*** ,
  - $A - B = \{x \mid x \in A \land x \notin B\}$

  Example: If ***A*** = {a , b , c , d } and B = {c , d , e , f } , then ***A - B*** = {a , b }.

  - The *intersection* is not commutative: $A - B \neq B - A$. .



# Difference Examples

- $\{1,2,3,4,5,6\} - \{2,3,5,7,9,11\} =$ $\underline{\qquad\{1,4,6\}\qquad}$
- $\mathbf{Z} - \mathbf{N} = \{\ldots, -1, 0, 1, 2, \ldots\} - \{1, 2, \ldots\}$
  $= \{x \mid x \text{ is an integer but not a natural}\}$
  $= \{\ldots, -3, -2, -1, 0\}$

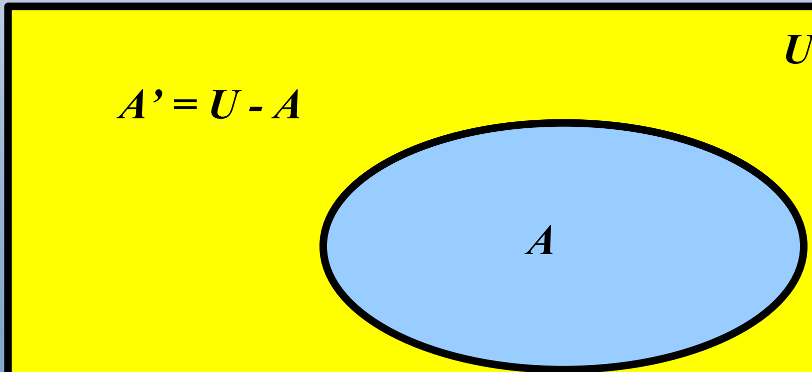
# Set Complements

- When the context clearly defines the universal set $U$, we say that for any set $A \subseteq U$, the *complement* of $A$, written $\overline{A}$ or $A'$ is the complement of $A$ with respect to $U$:

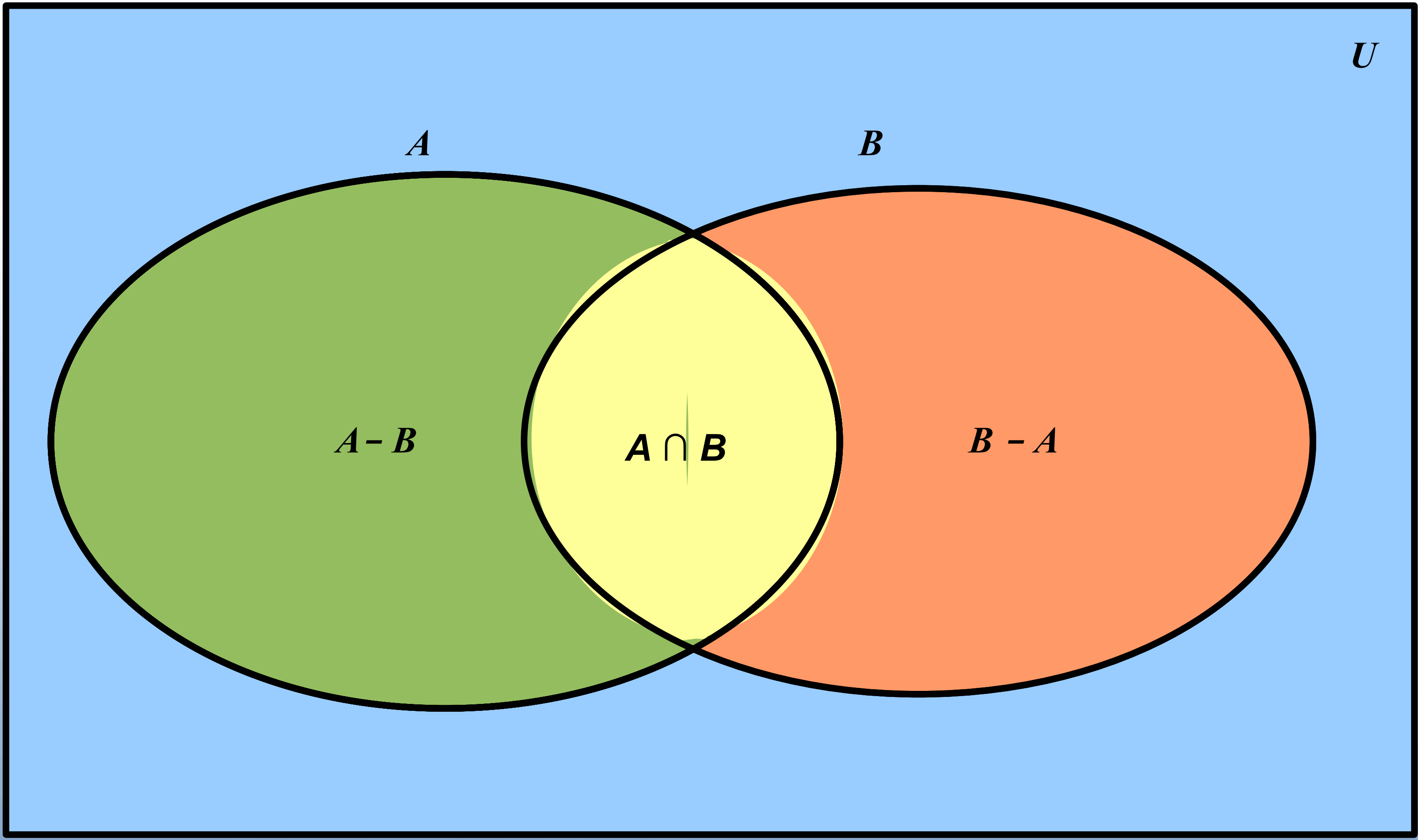
$$A' = U - A$$

Example: If $U = \mathbf{N}$, $A = \{3,5\}$

$A' = \{1, 2, 4, 6, 7\ldots\}$



# Basic Set Operations: summary



# Algebra of Sets (1)

## $U$ Universal set and its subsets $A$, $B$, $C$

The Identity Rules:

$$A \cup \varnothing = A$$

$$A \cap U = A$$

$$A \cup U = U$$

$$A \cap \varnothing = \varnothing$$

The Idempotent Rules:

$$(A')' = A$$

$$A \cup A = A$$

$$A \cap A = A$$

The Complement Rules:

$$A \cup A' = U$$

$$A \cap A' = \varnothing$$

$$U' = \varnothing$$

$$\varnothing' = U$$

# Algebra of Sets (2)

The Associative Rules:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

The Distributive Rules:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

The De Morgan Rules:

$$(A \cup B)' = A' \cap B'$$

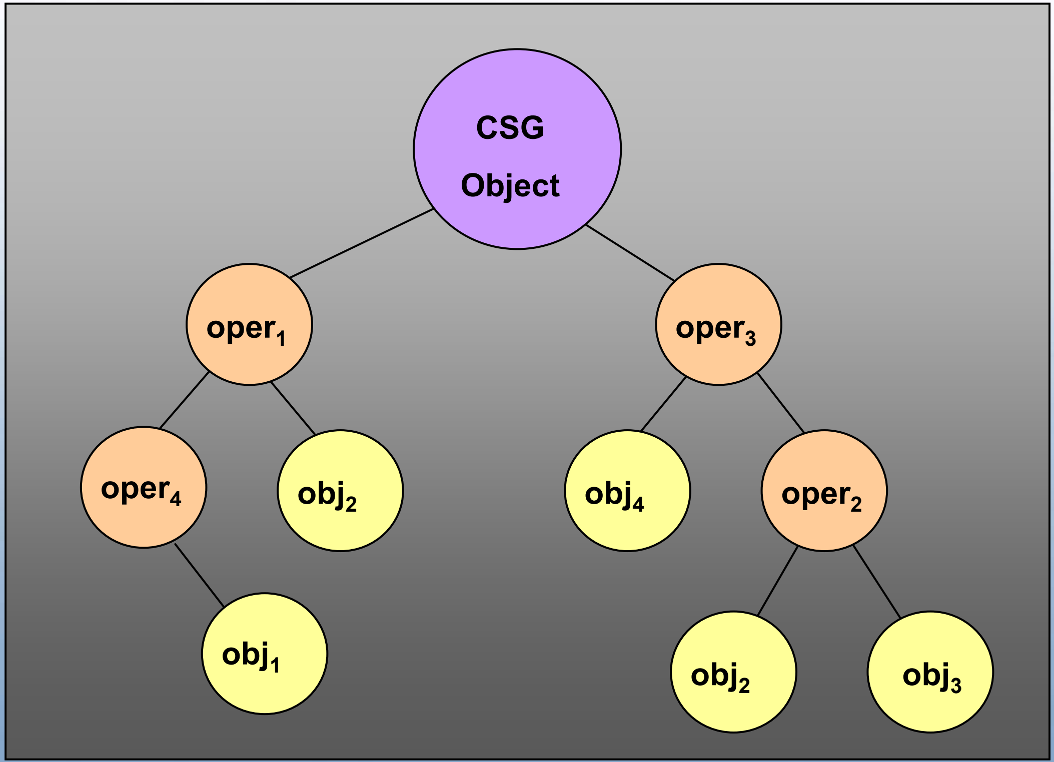
$$(A \cap B)' = A' \cup B'$$

# Constructive Solid Geometry (CSG)

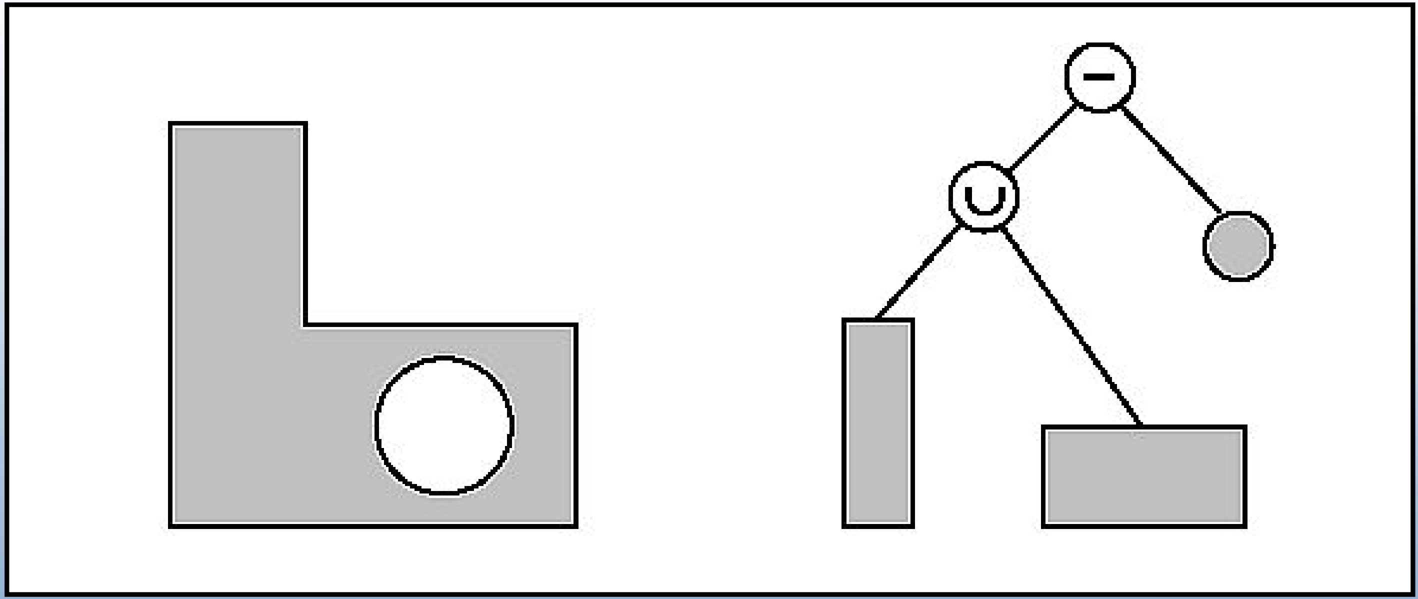
- CSG is based on a set of 3D solid primitives and set-theoretic operations
- Traditional primitives: block, cylinder, cone, sphere, torus
- Operations; union, intersection, difference + translation and rotation

# CSG tree

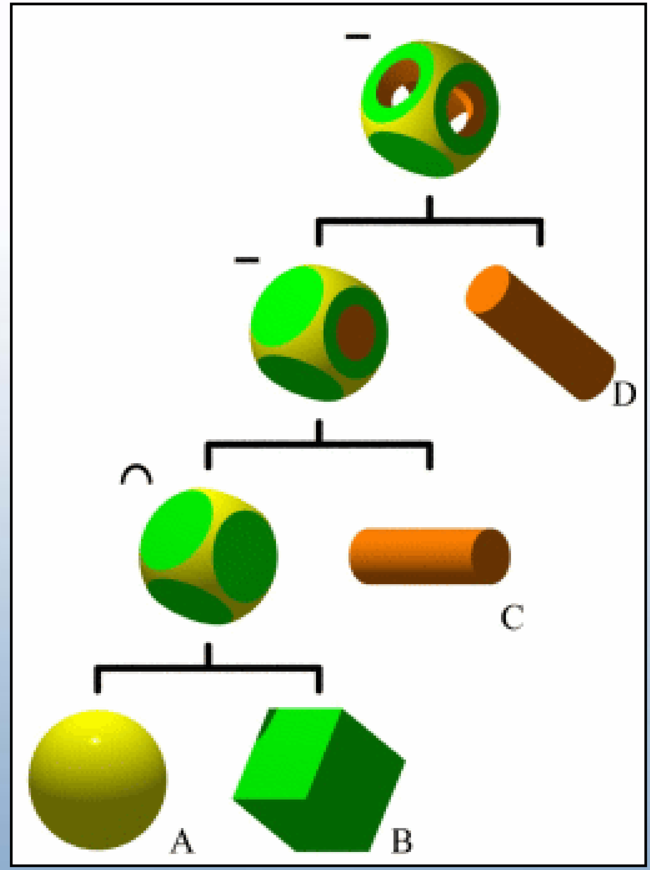
- A complex solid is represented with a binary tree usually called CSG tree



# *CSG tree of a 2D solid*



*CSG tree of a 3D solid*



# Cartesian Products of Sets

- For sets *A*, *B*, their *Cartesian product* $A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}$.

  is the set of all possible ordered pairs whose first component is a member of *A* and whose second component is a member of *B*

Example:

- $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$

- Other terms: product set, set direct product, or cross product
- If R is a *relation* between A and B then $R \subseteq A x B$

René Descartes (1596-1650)

# Cartesian Products of Sets

Example:

{John,Mary,Ellen}x{News,Soap} =

{(John,News), (Mary,News), (Ellen,News), (John,Soap), (Mary,Soap), (Ellen,Soap)}

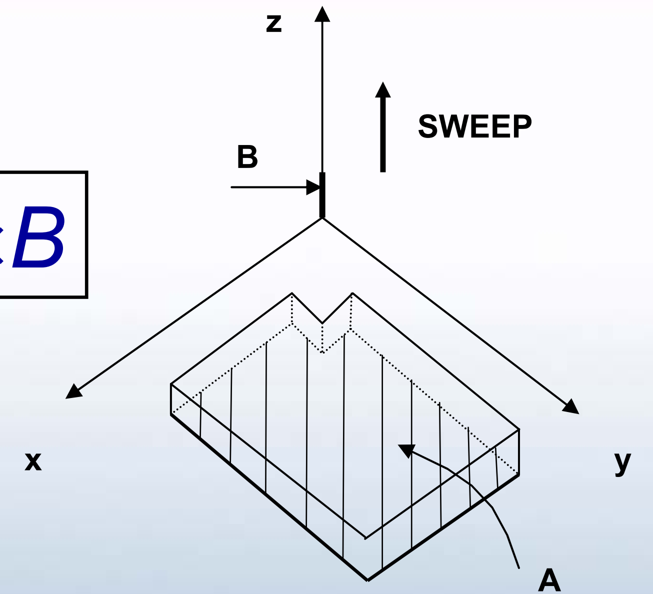
## Cartesian Products of Sets

- Note that
  - for finite $A$, $B$, $|A \times B| = |A|.|B|$
  - the Cartesian product is *not* commutative: $\neg\forall AB: A \times B = B \times A$.
  - notation extends naturally to $A_1 \times A_2 \times \ldots \times A_n$

# Sweep as Cartesian Product

- Set of all points visited by an object A moving along a trajectory B is a new solid, called a sweep.
- Translational sweeping (extrusion): 2D area moves along a line normal to the plane of the area.

$A \times B$



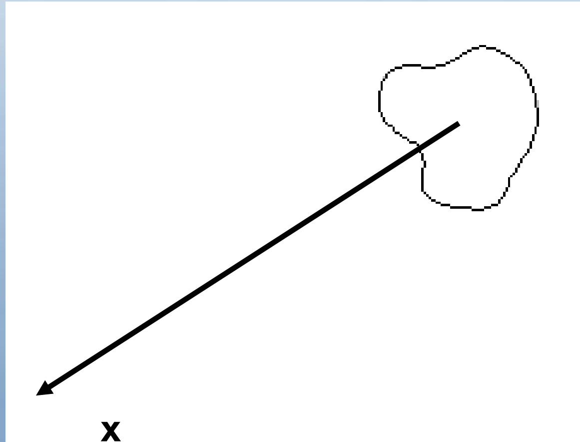




Image by Martin

# Review: Set Notations

- Set enumeration {a, b, c} and set-builder $\{x|P(x)\}$
- $\in$ relation, and the empty set $\emptyset$.
- Set relations =, $\subseteq$, $\supseteq$, $\subset$, $\supset$, $\not\subset$, etc.
- Cardinality $|S|$
- Power sets P($S$)
- Venn diagrams
- Set operations $\cup$, $\cap$, -, $\times$
- Constructive Solid Geometry, sweeping

# Contents


- Notions for sets
- Venn diagrams
- Basic set operations
- **Notions of functions**
- Properties of functions

# Functions

- From calculus, you know the concept of a real-valued function *f*, which assigns to each number $x \in \mathbf{R}$ one particular value $y=f(x)$, where $y \in \mathbf{R}$.
  - *Example*: *f* defined by the expression

$$f(x)=x^2$$

- The notion of a function can be generalized to the concept of assigning elements of *any* set to elements of *any* set.

# Function: Formal Definition

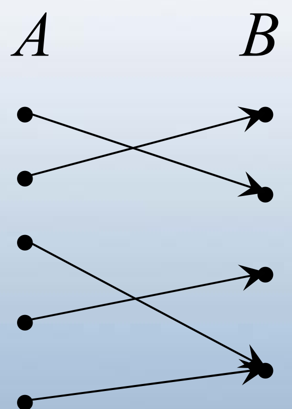
- For any sets $A$, $B$, we say that a *function f (or "mapping") from A to B* ($f{:}A\rightarrow B$) is a particular assignment of exactly one element $f(x)\in B$ to each element $x\in A$.
- Some further generalizations of this idea:
  - A *partial* (non-*total*) function $f$ assigns *zero or one* elements of $B$ to each element $x\in A$.
  - Functions of $n$ arguments; relations.

# Basic Properties of Functions

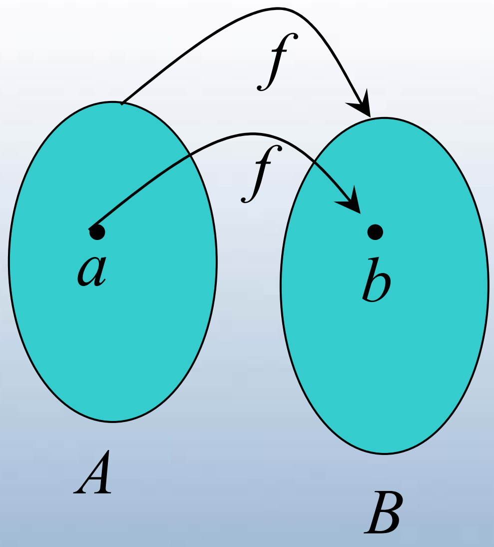
- We can represent a function $f\colon A \rightarrow B$ as a set of ordered pairs $f = \{(a, f(a)) \mid a \in A\}$.
- This makes $f$ a relation between A and B: f is a subset of A x B. But functions are special:
  - for every $a \in A$, there is at least one pair $(a,b)$. Formally: $\forall a \in A \exists b \in B((a,b) \in f)$
  - for every $a \in A$, there is at most one pair $(a,b)$. Formally: $\neg \exists a,b,c((a,b) \in f \wedge (a,c) \in f \wedge b \neq c)$

# Graphs of Functions

- Functions can be represented graphically in several ways:




Bipartite Graph

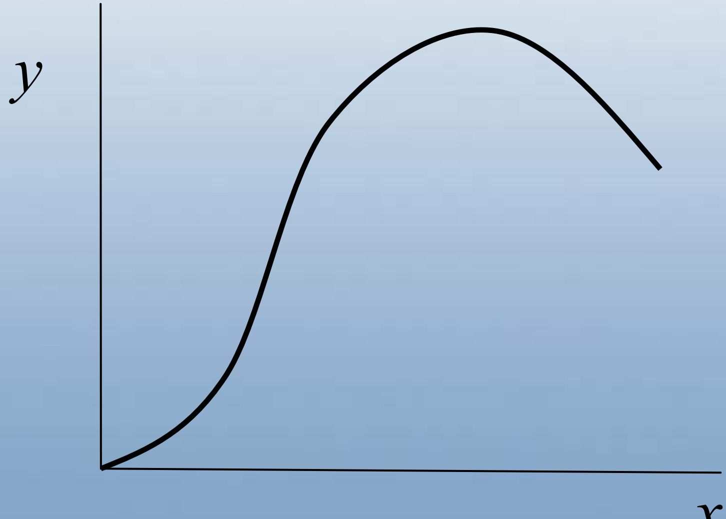



Like Venn diagrams

# Graphs of Functions

- A relation over numbers can be represented as a set of points on a plane. (A point is a pair *(x,y)*.)
- A function is then a curve (set of points), with only one *y* for each *x*.

Plot



# Some Function Terminology

- If $f$:$A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then we say:
  - $A$ is the *domain* of $f$.
  - $B$ is the *codomain* of $f$.
  - $b$ is the *image* of $a$ under $f$.
  - $a$ is a *pre-image* of $b$ under $f$.
    - In general, $b$ may have more than one pre-image.
  - The *range* $R \subseteq B$ of $f$ is $R=\{b \mid \exists a\ f(a)=b\}$.

# Range versus Codomain

- The range of a function may not be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

# Range vs. Codomain - Example

- Suppose I declare to you that: "*f* is a function mapping students in this class to the set of grades {A,B,C,D,E}."
- At this point, you know *f*'s codomain is: {A,B,C,D,E}, and its range is unknown.
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is {A,B}, but its codomain is still {A,B,C,D,E}.

# (n-ary) Functions on a Set

- An *n*-ary function (also: n-ary *operator) over S* is any function from the set of ordered *n*-tuples of elements of *S*, to *S* itself.
- Examples:
- if *S*={**T**,**F**}, $\neg$ can be seen as a unary operator, and $\wedge$,$\vee$ are binary operators on *S*.
- $\cup$ and $\cap$ are binary operators on the set of all sets.