National Centre for Computer Animation

2D Mathematical Foundations 3

Unfortunately, no one can be told what the Matrix is.
You have to see it for yourself.

ES Overview

- Introduction to ES
- **2D Graphics in Entertainment Systems**
- Sound, Speech & Music
- 3D Graphics in Entertainment Systems

Matrices

A matrix is a rectangular array (*table*) of scalars:

- its horizontal lines are called rows
- Its vertical lines are called columns

$$m = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \quad \text{or} \quad m = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

If the numbers of rows and columns are identical the matrix is a **square matrix**.

Matrix Operations

- Addition/Subtraction
- Multiplication
  - with a matrix
  - with a vector
- Transformations
  - Scale
  - Rotation
  - Translation

Matrix Addition

Matrices are added by adding their individual components.

$$o = m + n = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} m_{11} + n_{11} & m_{12} + m_{12} \\ m_{21} + n_{21} & n_{22} + n_{22} \end{bmatrix}$$

Matrices are subtracted by adding the inverse of the components.

Matrix Multiplication

Multiplying a matrix by a scalar value means to multiply all of the matrices components with that scalar:

$$m' = \mathrm{s}m = \mathrm{s}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} \mathrm{s} \cdot m_{11} & \mathrm{s} \cdot m_{12} \\ \mathrm{s} \cdot m_{21} & \mathrm{s} \cdot m_{22} \end{bmatrix}$$

Matrix Multiplication

Matrix multiplication (*scalar product*) is **not** commutative:

c = a · b ≠ c = b · a

- The number of columns of the left matrix must be equal to the number of rows of the right matrix (*in that case the matrices are called **conformant***).
- The resulting element c_{ij} is the dot product of the i^{th} row af a and the j^{th} column of b.

$$c = a \cdot b = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

≠

$$c = b \cdot a = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

Matrix Multiplication

For multiplying a matrix by a vector we treat the vector as a matrix with one column:

$$v' = m \cdot v = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 m_{11} + v_1 m_{12} \\ v_2 m_{21} + v_2 m_{22} \end{bmatrix}$$

It is important that the vector is on the right side of the multiplication operator if the resulting matrix is supposed to be a vector (*otherwise it will be a matrix*)

Identity Matrix

The identity matrix is a square matrix in which the values of its main diagonal are all 1 (*with all other values being 0*).

$$i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D square ID matrix

$$i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3D square ID matrix

Multiplying a matrix with the identity matrix has no effect.

Examples

- Multiplying two matrices **a** and **b**.

$$c = a \cdot b = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1\cdot5+2\cdot7 & 1\cdot6+2\cdot8 \\ 3\cdot5+4\cdot7 & 3\cdot6+4\cdot8 \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

- Multiplying matrix **a** with the identity matrix.

$$b = a \cdot i = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot1+2\cdot0 & 1\cdot0+2\cdot1 \\ 3\cdot1+4\cdot0 & 3\cdot0+4\cdot1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Multiplying a matrix **m** with a vector **v**.

$$v' = m \cdot v = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1\cdot5+2\cdot6 \\ 3\cdot5+4\cdot6 \end{bmatrix} = \begin{bmatrix} 5+12 \\ 15+24 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

Homogeneous Matrices

Some matrix operations require the use of a so called **homogeneous matrix**.

These require us to add an additional dimension to vectors and matrices (*which is left to resemble the identity matrix*):

$$m = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

normal 2D matrix.

$$m = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the same as a homogeneous matrix

Transformations

Transformations allow us to change vectors in space by multiplying them with so called transformation matrices.

These transformations include (*but are not limited to*):

- scaling
- rotating
- translating (*moving/shifting*)

of the vectors.

Transforms usually use homogeneous vectors & matrices.

Scaling

The matrix for 2D scaling (*with scaling factors* s_x *and* s_y) is:

non-homogeneous:

$$\vec{v} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

homogeneous:

$$\vec{v} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotating

The matrix for 2D rotation (*about the origin*) by angle θ is:

non-homogeneous:

$$\vec{v} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

homogeneous:

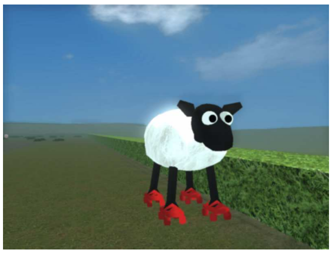
$$\vec{v} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translating

The matrix for 2D translation (*in directions t_x and t_y*) is always a homogeneous matrix:

$$\vec{v} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Matrices in Games



even sheep use matrices

- used for moving objects around on screen
- rarely used in 2D only environments
- essential for 3D computer graphics
  - in 3D practically everything depends on matrices