National Centre for Computer Animation

2D Mathematical Foundations 2

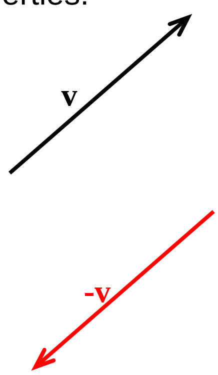
...And this is finite, where 'finite' is infinitely large.

Vectors

A vector is a geometric object with two properties:

- magnitude (*size/length*)
- direction

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$



This is encoded (*for 2D vectors*) in its two components, x and y (*corresponding to the coordinate system*)

The inverse of a vector v, -v is calculated by inverting each of its components (*-x and -y*).

Vector Operations

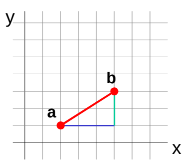
- Magnitude
- Addition/Subtraction
- Multiplication
  - With scalar & division by scalar
  - With vector (*dot product*)
- Normalisation

Vector Magnitude

Remember:

Using Pythagoras we can calculate the distance d between 2 points a & b:

$d = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2}$



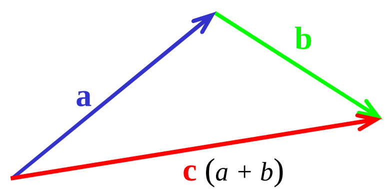
The magnitude of a vector v is : $|\mathbf{v}| = \sqrt{(v_x^2 + v_y^2)}$

- a vector with the magnitude 1 is called **unit vector**
- a vector with magnitude 0 is called **zero vector**

Vector Addition

Vectors are added by adding their components.

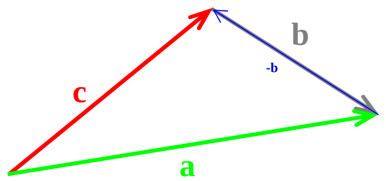
- **a + b = c** where ($c_x=a_x+b_x$ *and* $c_y=a_y+b_y$)
- it is commutative: **a + b = b + a**
- it is associative: **(a+b) + c = a + (b+c)**



Vector Subtraction

Vectors are subtracted by adding the inverse of the vector to be subtracted to the original vector:

- **a - b = c** ↔ **a + (**-b**) = c**

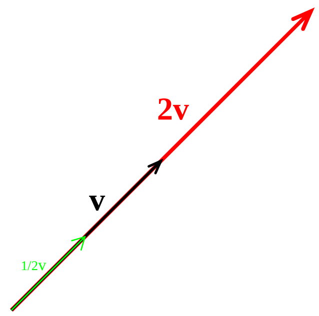


Vector Multiplication

A scalar has only a magnitude (*it is just a number*).

Multiplying a vector by a scalar value scales the vector (*changes the length of the vector*).

$$\frac{1}{2}\vec{v} = \begin{pmatrix} \tfrac{1}{2}x \\ \tfrac{1}{2}y \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad 2\vec{v} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

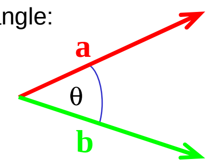


Vector Multiplication

- The scalar product of two vectors results in a scalar value.
- For the operator used it is also referred to as the dot product.

The dot product is calculated by multiplying the magnitudes of the vectors with each other and with the cosine of the enclosed angle:

$$a \cdot b = |a|\,|b| \cos(\theta)$$



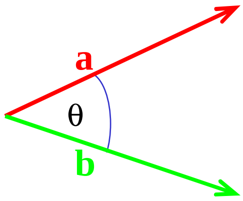
It can be derived from the sum of the products of the two vector's components:

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = a_x b_x + a_y b_y$$

Dot Product Usage

We can use the dot product to find the angle between two known vectors:

$$\cos(\theta) = \frac{a \cdot b}{|a||b|}$$



The inverse of the cosine then allows the identification of the enclosed angle.

- if the angle θ is 0 then both vectors are parallel
- If the resulting scalar is 0 then the vectors are perpendicular to each other

Vector Normalisation

Normalising a vector **a** means to make it into a unit vector (*of magnitude 1*).

This is achieved by scaling it down by its magnitude:

$$\hat{a} = a \;/\; |a|$$

Vectors & Points

Remember:

- A point is a location in space with an infinitely small extent.

The position of a point can be described by a vector whose components are identical to the coordinates of the point (*effectively a vector from the origin to the point*).
These vectors are called **position vectors**.

Both vectors and points are used in entertainment systems to represent geometric data (*using position vectors for encoding points*).

Vector Usage

In games vectors are used to encode

- Positions (*e.g. the vertices of a geometric model*)
- Directions (*of movement*)
- Forces (*applied to objects*)
  - including velocity/accelleration

Line (*parameterized*)

A line can be encoded using a point and a vector:

$$\vec{r} = \vec{a} + t\vec{b}$$

where

- **r** is a vector to any point on the line
- **b** is a vector parallel to the line (*pointing in the same direction*)
- **a** is the position vector of a point intersected by the line
- *t* is a real parameter

Line (*parameterized*)

Two lines $\mathbf{r}_1=\mathbf{a}_1+t\mathbf{b}_1$ and $\mathbf{r}_2=\mathbf{a}_2+s\mathbf{b}_2$ intersect if there are unique values for *s* and *t* and if $\mathbf{r}_1=\mathbf{r}_2$:

$$\mathbf{a}_1+t\mathbf{b}_1 = \mathbf{a}_2+s\mathbf{b}_2$$

This only works if both lines are not parallel

(*if the angle enclosed by the intersection in not 0 – see dot product*).